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1926

MATHEMATICS

FOR

STUDENTS OF AGRICULTURE

Part One

THE PENNSYLVANIA STATE COLLEGE

1926

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TO THE STUDENT

As a science becomes more and more exact, it becomes more and more mathematical. The new progress in the study of agriculture is making it more exact and scientific, and thus more mathematical. Hence, the need of a modern agriculturist for the exact methods of mathematics in all his work is steadily increasing. It is for this reason that students of agriculture are required to study mathematics.

For many years they have been required to study the traditional course in trigonometry. In recent years, however, there has been a general dissatisfaction with the results of such a formal study. It has been felt that another course might be designed which would stimulate the interest of the students by meeting their definite needs and dealing with problems of real life from their point of view.

This book aims to meet this need. The former work in trigonometry is reduced almost one half by eliminating the theoretical part which deals with identities and equations, and which is of chief value to students who wish to continue the study of higher mathematics. The remaining part of trigonometry is

developed in such a manner as not to suffer in logical treatment. More emphasis is laid on the applications to surveying and physics in order to prepare the student for his further studies in college and to meet the actual problems in after life.

The time gained from eliminating the purely theoretical part of trigonometry has made it possible to devote nearly half the semester to the study of topics more worth knowing. Particular attention is given to the study of graphical representation and statistical methods, since these subjects are steadily increasing in importance to an agriculturist if he is to appreciate scientific investigations and conclusions in the field of agriculture. The remaining chapters deal chiefly with the scientific formula and equation, a clear understanding of which should belong to the intellectual equipment of every college man. Throughout the whole book there is a large variety of problems from actual life situations in which men are vitally interested.

A glance at the various chapters is sufficient to show that the book is of college grade and requires college methods in order to develop the power of clear and logical thinking.

The course is still far from perfection; but it is believed to be a decided improvement on a formal course in trigonometry. Any suggestions or criticisms will be heartily welcomed.

Finally, the student should remember that every achievement that is worth while demands faithful work on his part. The text can do a great deal towards the success of a class; the teacher certainly has a contribution to make; but the chief part is played by the student. Unless he does his part well, text and teacher avail nothing. The text, the teacher, and the student, these three; but the greatest of these is the student.

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Part Two.

T R I G O N O M E T R Y

CHAPTER I

Equation and Formula

FUNCTIONS

In scientific work and in our daily life, we are constantly dealing with things that are related in some definite way. This relation is one of dependence. When a quantity depends upon another quantity for its value, it is said to be a function of the latter. Any change in the second produces some corresponding change in the first. It is important to study the exact relations between quantities and discover how they affect each other.

To illustrate: The price of wheat varies with the size of the crop. The area of a circle depends upon the length of the radius. The length of a rod depends upon its temperature. The speed of an automobile depends upon the amount of gas consumed. The interest on a sum of money changes as the rate of interest changes. The amount of sirloin steak that can be bought for a dollar depends upon the price per pound. The value of $x^2 - 2x + 3$ varies with the value of x, and so on.

See if you can illustrate the idea of function by ten familiar examples not given above.

The fact that we are always dealing with situations of this kind makes it necessary for us to know how to represent and determine these quantities which change together, or which are related in some definite way.

A powerful means of describing or expressing such relations is the EQUATION And the FORMULA.

The EQUATION

An equation is a statement that one expression is equal to another expression. It is the most important thing in mathematics for it is used to solve a very large number of problems in which unknown quantities must be found.

To solve an equation is to find the value of the unknown number or letter, in the equation which satisfies the equation, that is, which makes both sides of the equation equal. This value is called the solution or the root of the equation. The correctness of the solution is checked by substituting the obtained value of the unknown in the original equation, and noting whether the equation is satisfied, that is, whether the numerical values of the two sides balance.

The members of a class tried to solve the equation

$$8x - 4 = 28$$

Some decided that $x = 3$; while others said that $x = 4$. Which group was right?

How is the solution of an equation checked? What is the value of learning to check very carefully every kind of work you do?

Exercise

1. Is 1 a solution of $10x - 3 = 4 + 3x$?

2. Is 3 a solution of $6y + 10 = 10y + 2$?

3. Is 5c a solution of $x - 2c = 4c - x$?

4. Is -2 a solution of $x^2 + x - 6 = 0$?

5. Is -3 a solution of $x^2 + 5x + 6 = 0$?

6. Is 3 a solution of $2x^2 - 4x - 5 = 0$?

7. Is 12 a solution of $\frac{1}{2}y^2 - \frac{4}{3}y - 25 = 0$?

The Formula

A formula is an equation which expresses some practical rule from arithmetic, the shop, the trades, the sciences, the business world, etc. It is a short hand way of stating a rule. Thus

Rule _____ The area of a rectangle is equal to the product of its base and its altitude.

Formula _____ $A = a\ b$

Rule _____ To find the interest (expressed in dollars) multiply the principal (expressed in dollars) by the product of the rate (expressed in hundredths) and the time (expressed in years).

Formula _____ $I = p\ r\ t$

The symbolic form is not only more readily understood than the sentence form, but is more easily applied to a practical problem.

Observe that the formula is a special type of equation. It is an equation in which significant symbols are used to denote quantities. A significant symbol is one which by its form or by common use suggests the quantity for which it stands. Thus in the interest formula, I stands for interest, p for principal, r for rate, and t for time. The conventional equation would be $w = x\ y\ z$, no symbol of which suggests the quantity represented.

Since the formula is an equation, the same principles are applied to both

These principles are:

1. If equals are added to equals the sums are equal.

2. If equals are subtracted from equals the remainders are equal.

3. If equals are multiplied by equals(or by the same number) the products are equal.

4. If equals are divided by equals (or by the same number) the quotients are equal.

5. Numbers that are equal to the same number, or to equal numbers are equal to each other.

6. Like powers of equal numbers are equal.

7. Like roots of equal numbers are equal.

8. The whole of anything is equal to the sum of all its parts.

These principles are called axioms(assumptions). They operate on an equation or on a formula without disturbing the equality.

To solve an equation perform the indicated operations in the order of:

1. Those inclosed in parentheses.

2. Powers and roots.

3. Multiplications and divisions in order of occurrence.

4. Additions and subtractions.

Parentheses.

Parentheses are used for grouping. The expression they inclose should be considered as a single symbol. They indicate multiplication. Consider the equation

$$7 - 3(2x - 5) = 2 -(6 - 7x).$$

Notice that in the left member of the equation the -3 is not to be subtracted from 7. The expression in parentheses must be multiplied by -3. If no multiplier is written before the parentheses, as in -(6 - 7x), it is understood that the multiplier is 1. In this case it is -1. Performing the multiplication is better than the old method of removing the parentheses

To solve the equation we first multiply to re..ove the parentheses as follows:

$$7 - 6x + 15 = 2 - 6 + 7x$$

Transposing, gives

$$-6x - 7x = -7 - 15 + 2 - 6.$$

Collecting, $-13x = -26$

Dividing each side by - 13 gives

$$x = 2$$

Check: Substituting 2 for x in the original equation,

$$7 - 3(4 - 5) = 2 - (6 - 14)$$
$$7 - 12 + 15 = 2 - 6 + 14$$
$$10 = 10$$

Exercise.

Solve the following equations and check:

1. $4x - 3 = 2x + 7.$
2. $7y = 2y + 15.$
3. $13x - 5x = 2x - 20.$
4. $9 + x - 4x = 3x + 24.$
5. $10 - 6x = 14 - 8x.$
6. $5 - 2y + 10 = -2 + 5y.$
7. $y - 2(y - 5) = 0.$
8. $3(x - 4) - x = -20.$
9. $2a - 7(3 - a) = a + 8.$
10. $-(y - 4) - 3y = 2(y - 11).$
11. $5 - 2(y - 6) = -(4 - 5y).$
12. $x - (10 - x) - 10 = 4.$

EVALUATING A FORMULA.

The value of a formula depends upon the values given the letters. To find the value of a formula we proceed as follows:

1. Substitute the given numbers for their letters.
2. Reduce the result to the simplest form.

Exercise.

1. The area of a triangle with base b and altitude h is
   $A = \frac{1}{2}b\ h$.
   Find A if b = 4.5 and h = 3.2; if b = 10.3 and h = 7.7.

2. The hypotenuse of a right triangle. $c^2 = a^2 + b^2$.
   Find c if a = 8 and b = 6; if a = 16.5 and b = 12.3.

3. The area of a circle. $A = \pi r^2$.
   Find A if r = 8.3; if r = .72.

4. The area of a ring is $A = \pi(R^2 - r^2) = \pi(R - r)(R + r)$.
   Find A if R = 12.7 and r = 9.8; if R = 1.3 and r = .9.

5. The lateral surface of a right cylinder $S = 2\pi r\ h$.
   Find S if r = 3.4 and h = 11.2.

6. The volume of a cylinder $V = \pi r^2 h$.
   Find V if r = 6.1 and h = 15.
   What is the effect on V of doubling r? of doubling h?

7. The volume of a tube $V = \pi h(R^2 - r^2)$.
   Find V if h = 24, R = 3.5 and r = 2.8.

8. The volume of a cone is $V = \frac{\pi}{3} r^2 h$.
   Find V if r = $4\frac{1}{2}$ and h = 7 1/3.

9. The volume of a frustum $V = \frac{1}{3}\pi h(R^2 + r^2 + Rr)$
   Find the volume of a pail whose diameters are 12 and 15 and whose height is 12.

10. The surface of a sphere $S = 4\pi r^2$.
    Find S if r = $10\frac{1}{4}$; if r = .36.

11. The volume of a sphere $V = \frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3$.
    Find V if r = $5\frac{1}{2}$; if r = 3960; if d = 1.
    What is the effect on V of doubling r?

12. The volume of a barrel $V = \frac{1}{12}\pi h(2D^2 + d^2)$,
    Where D = bung diameter, d = head diameter, and h = height.
    Find V if h = $3\frac{1}{2}$, D = 2 and d $1\frac{1}{2}$.

13. The area of a triangle in terms of its sides, a, b, c.
    $A = \sqrt{s(s - a)(s - b)(s - c)}$, where $s = \frac{1}{2}(a + b + c)$.
    Find A when the sides are , 3.4, 4.2, 5.6.

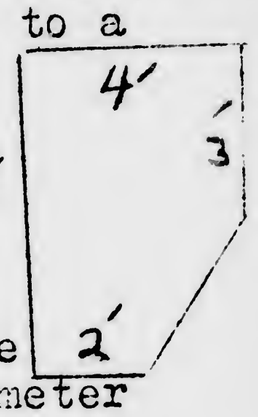
14. The area of a parallelogram with base b and height h.
A = b h. Find A if b = 15.3 and h = 8.2

15. The area of a trapezoid with bases b and b′.
A = $\frac{1}{2}(b + b')h$. Find A if b = $16\frac{1}{2}$, b′ = 10 and h = 6.

16. What would be the size of a water spout to be equivalent to two spouts 3 in. and 4 in. in diameter?

17. How many revolutions per minute must a drive wheel 4 feet in diameter make to give a speed of 1884 r. p.m. to a shaft-pulley 8 inches in diameter?

18. How many bushels of wheat does a bin hold if it is 10 feet long and its cross section is as shown in the figure?



19. How much concrete is there in a circular silo whose walls are 10 inches thick, 14 ft. outside diameter and 25 ft. high? How many cubic feet of silage does it hold?

Note on the Silo. When silage is left exposed to the air for any length of time it spoils. To prevent spoiling it is necessary to remove a layer of silage to a depth of not less than 2 inches daily. The diameter of a silo must, therefore, be planned according to the number of cattle to be fed, while the height is determined by the quantity to be fed. A cubic foot of silage averages 40 pounds. The daily ration per head averages 40 pounds.

20. What should be the dimensions of a round silo for a herd of 24 cows, if each one is to be fed 40 pounds daily for 160 days? (Allow three feet for setting of silage). (Ans. Diam. = 13.54 ft; H = 29.7 ft.)

21. What should be the height of a round silo for a herd of 20 cows if each one is to be fed 40 pounds daily for 150 days?

22. What is the capacity of a round silo 40 ft. high and 12 ft. in diameter? How many cows will it maintain? How many pounds daily can each be fed from this silo for 180 days?

23. Fattening beef cattle should be fed about 25 pounds a day. What should be the dimensions of a round silo if 2 inches of depth are fed daily to a herd of 50 cows for 100 days?

24. Some authorities estimate that 5 square ft. of horizontal feeding should be allowed for each cow. What should be the diameter of a silo for a herd of 30 cows? For a herd of 40 cows?

Chapter II.

P R O D U C T S A N D F A C T O R I N G

Some expressions occur so frequently in mathematics that a mastery of them will save a great deal of time. Among these are the following types of products and factoring. The student should learn them backward and forward. His success will depend upon his ability to determine the type to which an expression belongs from its appearance.

The relation of factoring to multiplication and division should be kept in mind. In multiplication we have two factors given, and we are required to find their product. In division we have the product and one factor given, and we are required to find the other factor. In factoring, however, we have only the product given and are required to find the factors. The results of factoring may be checked by actual multiplication.

Type Forms.

(1) $ax + ay - az = a(x + y - z)$.

Example $3x^3 - 15x^2 + 9x = 3x(x^2 - 5x + 3)$.

(2) $a^2 - b^2 = (a - b)(a + b)$.

$16x^4 - 25y^2 = (4x^2 - 5y)(4x^2 + 5y)$.

(3) $a^2 + 2ab + b^2 = (a + b)(a + b) = (a + b)^2$.

$x^2 + 8xy + 16y^2 = (x + 4y)^2$.

(4) $a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2$.

$49x^2 - 14xy + y^2 = (7x - y)^2$

(5) $x^2 + (a + b)x + ab = (x + a)(x + b)$.

$(x^2 - 4x - 45) = (x - 9)(x + 5)$.

(6) $acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$.

$18x^2 + 21x - 15 = 3(2x - 1)(3x + 5)$.

(7) $ax + ay + bx + by = a(x + y) + b(x + y) = (a + b)(x + y)$

$c^2x + c^2y - x - y = c^2(x + y) - (x + y) = (x + y)(c^2 - 1)$

$= (x + y)(c - 1)(c + 1)$.

Exercise.

Find the following products mentally:

1. $5x^2(1 - 3x)$
2. $(1 - 2y)^2$
3. $(2x + 5)(2x - 5)$
4. $(y - 7)(y + 3)$
5. $(x + \frac{1}{2})^2$
6. $(5x^2 - \frac{1}{4}y)^2$
7. $(3r + 8)(r - 6)$
8. $(1 - 2x)(4 + 5x)$
9. $(3x - \frac{1}{2})(3x + \frac{1}{2})$
10. $(2x + y - z)^2$

Find the prime factors of the following expressions:

11. $p + prt$
12. $6x^2 - 18x^3$
13. $r^4 + 7r^3 + 12r^2$
14. $9 - 36x^2$
15. $16a^4 - b^4$
16. $y^2 - y - 42$
17. $2x^2 - 6x - 20$
18. $9a^2 - 12ab + 4b^2$
19. $\pi R^2 - \pi r^2$
20. $2\pi r^2 + 2\pi rh$
21. $x^2 + 2xy + y^2 - z^2$
22. $9x^2 - \frac{1}{4}$
23. $x^2 + 6x + 8$
24. $4x^2 + 4x + 1$
25. $ax - ay + bx - by$
26. $9m(m - 2n) + 4n(m - 2n)$
27. $(48)(52)$
28. $1 - m^2 + 2mn - n^2$

Solve each of the following for x:

29. $x(a + 2) = a^2 - 4$
30. $x(2 + a) = 6 + 5a + a^2$
31. $xm - x = m^2 - 4m + 3$
32. $4d^2 + 2dx - c^2 = -cx$
33. How many terms do you get when you square the sum of two numbers? $(2a + 3b)^2 = ?$
34. Does $(x + y)^2 = x^2 + y^2$? Does $(x - y)^2 = x^2 - y^2$? or $x^2 + y^2$? Illustrate your answers by substituting numbers for x and y.
35. $x^2 + 8x + K$ is a perfect square; find K.
36. $x^2 - Kx + \frac{1}{4}$ is a perfect square; find K.
37. The area of a rectangle is $2x^2 + 5x + 3$. Find expressions for its dimensions.

Chapter III.

F R A C T I O N S

A fraction is an indicated division. It means the quotient of the numerator by the denominator. Thus $\frac{x}{y}$ means "x divided by y".

The principles in handling fractions are the same in algebra as in aithmetic. The following principles should be thoroughly understood and learned;

1. To multiply both numerator and denominator of a fraction by the same expression does not change the value of the fraction.

2. To divide both numerator and denominator by the same expression does not change the value of the fraction. Remember that dividing the numerator or denominator means dividing every term in it.

3. To reduce a fraction to its lowest terms factor both numerator and denominator and then divide out the common factors if there are any.

4. If the degree of the numerator of a fraction equals or exceeds that of the denominator, in a given letter, the fraction may be reduced to an integer or a mixed expression by dividing the numerator by the denominator.

5. In order to add or subtract fractions, they must be reduced, as in arithmetic, to equivalent fractions which have the same denominator. Then, write in succession over the common denominator, the numerators of the equivalent fractions, inclosing each numerator in parentheses preceded by the sign of the corresponding fraction. Then reduce this result into its simplest form.

6. To multiply a fraction by an integral expression, multiply the numerator by the integral expression and write the result over the denominator.

7. To multiply a fraction by a fraction, multiply the numerators together for a new numerator and the denominators together for a new denominator. Reduce the fraction to simplest form.

8. To divide a fraction by a number, divide the numerator by that number or multiply the denominator by that number.

9. To divide a fraction by another invert the divisor, as in arithmetic,and multiply.

10. Dividing the numerator of a fraction by a number gives the same result as multiplying the denominator by that number.

11. In every fraction there are three signs to be considered, the sign of the fraction, the sign of the numerator and the sign of the denominator,

$$-\frac{+a}{+b} = +\frac{-a}{b} = +\frac{+a}{-b} = -\frac{-a}{-b}.$$

Similarly, $+\frac{2x-1}{x-4} = -\frac{+1-2x}{x-4} = -\frac{2x-1}{4-x} = +\frac{+1-2x}{4-x}$

The sign of the numerator and the sign before the fraction can be changed without altering the value of the fraction. Changing the sign of the numerator means changing the sign of every term in the numerator. In $\frac{x}{4} - \frac{3x+2}{x}$, note that - is not the sign of 3x, but the sign of the fraction. The sign of 3x is + .

12. The reciprocal of a number is 1 divided by that number. The product of two reciprocals is 1. The reciprocal of a fraction is the fraction inverted.

13. If two fractions are equal their reciprocals are equal.

14. Any integer may be regarded as a fraction with denominator 1.

15. If zero is multiplied by a number, the product is zero. $0 \cdot a = 0$; $0 \times 5 = 0$.

16. If zero is divided by a number, the quotient is zero. $\frac{0}{a} = 0$; $\frac{0}{3} = 0$; $\frac{0}{-1} = 0$.

17. Division by 0 is not allowed. $\frac{a}{0}$ is meaningless. Sometimes it is written $\frac{a}{0} \to \infty$ (infinity).

This is to indicate that when the denominator of a fraction decreases and approaches zero, the value of the fraction increases without limit.

18. $\frac{0}{0}$ is undetermined . $\frac{0}{0} = 4$, 10, -1, 0, etc. (Why?)

19. In solving equations or formulas which involve fractions, multiply both sides of the equation(or formula) by a convenient expression to clear the fractions. The most convenient expression is the least common multiple of all the denominators.

Exercise.

1. Is the value of a fraction changed by adding the same expression to its numerator and denominator? Illustrate.

2. Is the value of a fraction changed by subtracting the same number from its numerator and denominator? Illustrate.

3. Do you change the value of a fraction by squaring both numerator and denominator? Illustrate.

4. Is the value of a fraction altered if the sign of the fraction and the sign of the denominator are both changed? What is meant by changing the sign of the denominator?

5. How does an increase in the numerator affect the value of a fraction? A decrease?

6. How does an increase in the denominator affect the value of a fraction? A decrease?

Find the algebraic sum of:

7. $\frac{5}{12} - \frac{7}{18} + 1$

8. $\frac{3x - 5}{4} - \frac{2 - x}{6} + \frac{7x}{8}$

9. $\frac{3x + 1}{5x} - \frac{4x + 3}{3x^2} - \frac{5 - x}{6x}$

10. $5x + 7 - \frac{25x^2}{5x + 7}$

11. $\frac{2x + 1}{x^2 - 9} - \frac{4}{x - 3}$

12. $b - \left(1 - \frac{2c + b}{c - b}\right)$

13. $\frac{4x}{x^2 - 1} + \frac{3}{x^2 + 2x + 1}$

14. $\frac{1}{x} + \frac{1}{y} - \frac{1}{x + y}$

15. $3 + \frac{5a}{a - 7} + 2a$

16. $x - \frac{x - 6}{2x - 3} - 2$

17. $\frac{3y - 1}{y^2 - 2y - 3} - \frac{3}{y - 3}$

18. $2 - \frac{4(x + y)}{x - y} + \frac{(x + y)^2}{(x - y)^2}$

19. $\frac{a^2 - 16a}{4 - a^2} - \frac{3 + 2a}{a - 2} + \frac{3a - 2}{a + 2}$

20. $\frac{1}{3(x + 1)} - \frac{x + 2}{3(-4 - 3x + x^2)}$

Reduce the following to integral or mixed expressions:

21. $\frac{4a^2 - 6ab + 3b^2}{2a - b}$

22. $\frac{x^3 + 3x^2y - 6xy^2 + 2y^3}{x^2 + 2xy + y^2}$

23. $\frac{x^2 + 4xy + y^2}{x + y}$

24. $\frac{x^3 - 8y^3}{x - 2y}$

Perform the indicated operations:

25. $\frac{5}{6} \cdot \frac{12}{7} \cdot 3$

26. $(y^2 + 6y + 9) \cdot \frac{y + 2}{y + 3}$

27. $\frac{x^2 - 4y^2}{a^2 - b^2} \div \frac{x - 2y}{a + b}$

28. $\frac{x^2 + x - 30}{x^2 + 5x - 6} \div \frac{x^2 - 25}{x^2 - 6x + 5}$

29. $\left(\frac{-c}{2d}\right)^2 \div \left(\frac{-d}{c}\right)^2 \cdot (-c)^3$

30. $\left(3y - 5 - \frac{168}{9y + 15}\right)\left(3 - \frac{4}{3 + y}\right)$

31. $1 \div \left(\frac{x}{y} + \frac{y}{x}\right)$

32. $\left(\frac{a}{b} - \frac{b}{a}\right) \div \frac{a - b}{a + b}$

33. $\left(1 - \frac{x}{y} + \frac{x^2}{y^2}\right) \div \left(1 + \frac{x}{y} + \frac{x^2}{y^2}\right)$

34. $\frac{ay + ax}{b} \div \frac{y^3 + x^3}{by - 3b} \cdot \frac{y^2 - xy + x^2}{y^2 - 5y + 6}$

35. Is $\frac{a + b}{x} = \frac{a}{x} + \frac{b}{x}$? Is $\frac{x}{a + b} = \frac{x}{a} + \frac{x}{b}$?
Illustrate by numbers.

36. Is $\frac{x + a}{x + b} = \frac{a}{b}$? This is the case of the Cancellatio Asinorum.

Complex Fractions.

A complex fraction is one that contains a fraction either in the numerator, or the denominator, or in both. A complex fraction may be simplified by multiplying every term in the main numerator and every term in the main denominator by the least common multiple of the denominators of the sub-fractions.

$$\frac{a + \frac{b}{x}}{\frac{c}{y} + \frac{d}{z}} = \frac{xyz\left(a + \frac{b}{x}\right)}{xyz\left(\frac{c}{y} + \frac{d}{z}\right)} = \frac{axyz + byz}{cxz + dxy}.$$

This is shorter than reducing the numerator and denominator separately and then inverting and multiplying. Even the middle step in the above example need rarely be written.

Example. $$\frac{1 + \frac{8}{x} - \frac{9}{x^2}}{x - 3 + \frac{2}{x}} = \frac{x^2 + 8x - 9}{x^3 - 3x^2 + 2x} = \frac{x + 9}{x^2 - 2x}$$

To check we select a small number and substitute for the unknown in the original expression and the final result to see if they are equal. Let $x = 3$.

$$\frac{1 + \frac{8}{3} - \frac{9}{9}}{3 - 3 + \frac{2}{3}} = \frac{3 + 9}{9 - 6} \quad \text{or} \quad 4 = 4.$$

2 and 1 and 0 cannot be used here because they make some denominator zero, and this must be avoided.

Exercise.

Simplify the following:

1. $\dfrac{3 + \frac{2}{5}}{4 - \frac{1}{3}}$.

2. $\dfrac{\frac{a}{b}}{\frac{c}{}} \div \dfrac{a}{\frac{b}{c}}$.

3. $\dfrac{\frac{8}{y^2} - y^2}{1 + \frac{y}{2} + \frac{2}{y}}$.

4. $\dfrac{\frac{x}{y} - 1}{2x - \frac{x^2 + y^2}{y}}$.

5. $\dfrac{x}{1 - \frac{y}{x}} + \dfrac{y}{1 - \frac{x}{y}}$.

6. $\dfrac{a^2 - \frac{1}{a}}{a + \frac{1}{a} + 1}$.

7. $\dfrac{\frac{1}{1 + a} + \frac{1}{1 - a}}{\frac{1}{1 - a} - \frac{1}{1 + a}}$.

8. $2 + \dfrac{3}{1 + \frac{2}{5}}$

9. $\dfrac{\frac{x + y}{x - y} - \frac{x - y}{x + y}}{\frac{x + y}{x - y} + \frac{x - y}{x + y}}$.

10. $\dfrac{1 - \frac{b}{a + b}}{1 + \frac{b}{a - b}}$.

11. $\dfrac{-3y - (y - 2x)\left[\frac{2y - 4x}{2y - x} - 1\right]}{(2y - x)^2}$.

12. $\dfrac{2 - 4y - 2x\left(-\frac{4x}{1 - 2y}\right)}{(1 - 2y)^2}$.

Equations Involving Fractions.

In solving equations which involve fractions, multiply both sides of the equation by the least common multiple of the denominators in order to clear fractions. Especial care is required to avoid error when a fraction is preceded by a minus sign. Then solve as usual and check.

Exercise.

Solve the following equations and check. In checking find the value of each fraction separately:

1. $\frac{c}{5y} = \frac{-2}{y-3}$.

2. $\frac{x+3}{x-4} = \frac{x+5}{x-16}$.

3. $\frac{4}{x-10} - \frac{3}{x-8} = 0$.

4. $\frac{3x-2}{4} - 3 - \frac{1-3x}{12} = \frac{2x-1}{3}$

5. $\frac{x+5}{x-8} - \frac{7x+3}{x^2-2x-48} = \frac{x+2}{x+6}$.

6. $\frac{3x}{2} + \frac{x+7}{3-x} = x$.

7. $\frac{y}{y-3} + 3 = \frac{1}{3-y}$.

8. $.06y - 4.5 = 1.68$.

9. $\frac{.7a}{4} - \frac{.1a}{6} = \frac{.19}{3}$.

10. $\frac{2-1.8y}{1.7} = \frac{.7y-5.7}{1.8}$.

Solving a Formula for Any One of its Letters.

A formula usually stands solved for one letter in terms of several others. For example, $i = p\,r\,t$. Here i is stated in terms of p, r, and t.

It often happens that one wishes to express p in terms of i, r, and t. To do this it is only necessary to solve the formula as an equation, and find the value of the particular letter desired in terms of the others. Thus,

$$i = p\,r\,t, \quad p = \frac{i}{rt}, \quad r = \frac{i}{pt}, \quad t = \frac{i}{pr}$$

are various forms of the same formula. It is only necessary to remember one of these forms and to be able to derive the others. This process is sometimes called "changing the subject", or "transformation of the formula".

Following are the general principles by means of which most formulas may be solved for any required quantity. These principles are based on the axioms. The quantity for which solution is required is called the unknown quantity;

1. If the unknown is under a radical, make the radical the only quantity in one of the members of the formula, and then raise both sides of the equation to a power corresponding to the root index.

2. Remove parentheses by the method of multiplication.

3. If the unknown is in the denominator of a fraction, multiply both sides of the equation by that denominator.

4. Collect all the unknown terms into the first member of the equation and all other terms into the second, as in the solution of an equation.

5. When the terms have been collected, divide both sides of the equation by the coefficient of the unknown with whatever sign it happens to have. (Sometimes it is necessary to take the unknown as a common factor of the terms containing it).

Exercise.

1. The formula $v = \sqrt{2gh}$ gives the velocity in terms of the height. Solve for h and find the height to which a stone will go if thrown upward with a velocity of 200 feet per second. ($g = 32.2$)

Squaring both sides, $v^2 = 2gh$

Dividing both sides by $2g$, $h = \frac{v^2}{2g}$

Substituting, $h = \frac{200^2}{2 \times 32.2} = 621.1$

2. The formula $v = v_0 + 32.2t$ gives the velocity of a falling body. Solve for t and find t if $v = 211$ and $v_0 = 50$.

3. $s = ct + \frac{1}{2}gt^2$. Solve for c. Find c when $s = 660$, $t = 4$ and $g = 16.1$.

4. The law of the pendulum is $t = \pi\sqrt{\frac{L}{g}}$.

Solve for L and find L when $t = 1$; when $t = 2$.

5. $C = \frac{E}{R + r}$. Solve for r. Find the value of r when C = 15.6, R = 4.3 and E = 110.7.

6. $\frac{E}{c} = \frac{R + r}{r}$. Solve for r. Find the value of r when E = 7.6, c = 2.3, R = 52.7.

7. $R = \frac{gs}{g + s}$. Solve for s in terms of R and g. Find s when R = 2.75 and g = 3.9.

8. $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$. Solve for F.

9. Temperature Fahrenheit in terms of Centigrade.
$F^0 = \frac{9C^0}{5} + 32^0$. Solve for C. Find C when $F = 212^0$.

10. Law of Inclined Plane.
$\frac{P}{W} = \frac{h}{L}$. Solve for h and L,

11. Law of diameter of shafting.

$d = \sqrt[3]{\frac{50\ H}{N}}$. Solve for H. Find H if d = 2, N = 100

12. The sum of an arithmetic progression.
$s = \frac{1}{2}n(a + L)$. Solve for a; for L.
Find a if L = 17, n = 10 and s = 100.

13. $L = a + (n - 1)d$. Solve for a; for d.

14. The sum of a geometric progression.
$s = \frac{rL - a}{r - 1}$. Solve for a; for r; for L.

15. Find r if s = 2046, a = 2, L = 512.

15. The total area of a prism.
$T = ph + 2A$. Solve for p, for A.

16. The volume of a solid ring.

$V = 2\pi R \cdot \pi r^2 = 2\pi^2 R\, r^2$.
Solve for r. Find r when V = 60 and R = 5.

17. The law of cosines.
$a^2 = b^2 + c^2 - 2bc \cdot \cos A$. Solve for cosA. Find cosA if a = 16.2; b = 8.3, c = 11.

18. The amount for simple interest.
$A = P(1 + rt)$. Solve for each letter.
At what rate will \$520 amount to \$600 in 3 years?

What principal will amount to \$1000 in 3 years 6 months at 6%?

CHAPTER IV

R A T I O A N D P R O P O R T I O N.

The proportional relation of numbers is very common in ordinary life, in industry, and in the sciences. In this section we shall study the mathematics of such relations.

Quantities are sometimes compared by finding how much one quantity is larger or smaller than another. For example, to compare a pole 10 ft. long with another pole 6 ft., we say the first pole is 4 ft. longer than the second; or the second pole is 4 ft. shorter than the first.

Ratio. Another way of comparing two quantities is to divide one quantity by the other. The quotient is called their ratio.

Definition. The ratio of one number to another is the quotient obtained by dividing the first by the second. Thus the ratio of 10 to 6 is $\frac{10}{6} = \frac{5}{3}$, while the ratio of 6 to 10 is $\frac{6}{10} = \frac{3}{5}$. The ratio of a to b is $\frac{a}{b}$. This is also written a : b.

It follows that all ratios are fractions and all fractions may be regarded as ratios. Hence the same laws apply to fractions and ratios. For example, the value of a ratio is not changed by multiplying or dividing both quantities by the same number. Thus

$$\frac{a}{b} = \frac{2\,a}{2\,b} = \frac{ax}{bx} = \frac{a \div x}{b \div x}.$$

In order to have a ratio the quantities compared must be of the same kind and must be expressed in terms of the same unit. The ratio of 2 ft. to 4 in. is $\frac{2 \times 12}{4} = 6$. Obviously there is no ratio between quantities of different kinds as 6 books and 2 bushels.

It is clear that to measure a quantity is to find its ratio to a given unit of measure. Thus, when we say the table is 4 ft. long, we mean that it is 4 times as long as the standard unit, the foot; that is, the ratio is 4 to 1.

Proportion. A proportion is an equation which states that two ratios are equal as $\frac{9}{12} = \frac{3}{4}$. The proportion $\frac{a}{b} = \frac{c}{d}$ is sometimes written a : b = c : d, and is read "a is to b as c is to d".

Since a proportion is an equation, any operation which may be performed on an equation may be performed on a proportion.

Means and Extremes. The first and last terms in a proportion are called the extremes, and the second and third terms are called the means. In the proportion $\frac{a}{b} = \frac{c}{d}$, a and d are the extremes, and b and c are the means.
If we multiply both sides of the equation by bd, we get ad = bc, which proves the

Principle. In any proportion the product of the means is equal to the product of the extremes. This principle is a convenient test of a proportion.

Mean Proportional. When the two means of a proportion are the same number, that number is called a mean proportional between the extremes, as b in $\frac{a}{b} = \frac{b}{c}$. Since $b^2 = ac$, $b = \pm\sqrt{ac}$.

Proportions From Equal Products. If the product of two numbers is equal to the product of two other numbers, either pair may be made the means and the other pair the extremes of a proportion.

For example, If ad = bc,
then $\frac{a}{b} = \frac{c}{d}$, or $\frac{a}{c} = \frac{b}{d}$, or $\frac{b}{a} = \frac{d}{c}$, or $\frac{c}{a} = \frac{d}{b}$.

Thus a proportion may be arranged in four different forms. Test these forms by the method of the products of the means and the extremes.

Similarity. Figures which have the same shape are called similar. Two figures are similar if the corresponding angles are equal and the corresponding sides are proportional.

It is proved in geometry that if two triangles (or polygons) are mutually equiangular, the ratios of the corresponding sides are equal(that is, the sides are proportional).

Exercise.

1. The sides of a triangle are 5,8, and 10. The longest side of a similar triangle is 14. What are the other sides?

2. In an architect's plan a wall 42 ft. long is represented by a line $10\frac{1}{2}$ in., what is the height of a window represented by $1\frac{1}{4}$ in.?

3. In the triangle A B C, the line D E is drawn parallel to A B. Show that $\frac{C\ D}{C\ A} = \frac{C\ E}{C\ B}$ and that $\frac{C\ D}{D\ A} = \frac{C\ E}{E\ B}$.

4. Show that the perpendicular from any point on a circle to a diameter is a mean proportional between the segments of the diameter.

5. The shadow of a yardstick is 1 ft. 3 in. How high is a pole whose shadow is 20 ft.?

6. A rod 15 ft. long is divided into two parts whose ratio is $\frac{2}{5}$. Find the length of each part.

7. If one cubic foot of lime and 2.5 cu. ft. of sand are used in making 2.9 cu. ft. of mortar, how much of each is needed to make 64 cu. ft. of mortar?

8. To find what per cent 18 is of 64, write $\frac{18}{64} = \frac{x}{100}$, and solve this proportion.

9. Find a mean proportional between 2 and 12; between a and b.

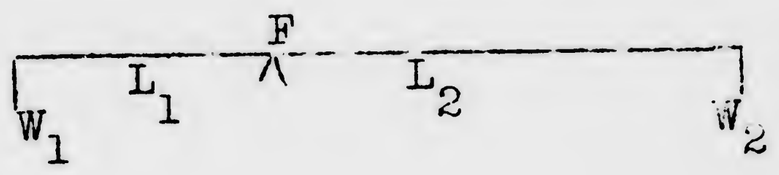
10. If $\frac{a}{b} = \frac{c}{d}$, show that

(a) $\frac{a+b}{b} = \frac{c+d}{d}$ (b) $\frac{a+b}{a} = \frac{c+d}{a}$

(c) $\frac{a-b}{b} = \frac{c-d}{d}$ (d) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

11. We have from physics that any lever is balanced when
Weight × weight arm = Force × force arm
that is $W_1 \cdot L_1 = W_2 \cdot L_2$

or $\frac{W_1}{W_2} = \frac{L_2}{L_1}$.



12. A and B are 4 ft. and 5 ft. respectively from the fulcrum. If B weighs 75 lb., how much does A weigh?

13. A lever 12 ft. long carries weights of 50 lb. and 70 lb. at its ends. Where should the fulcrum be placed so as to make the lever balance?

14. What portion of $\frac{8}{9}$ is $\frac{3}{4}$?

15. What is the best way of finding whether or not two fractions are equal? Is $\frac{22}{37} = \frac{28}{43}$?

16. Prove that the areas of two circles are proportional to the squares of their radii.

17. Prove that the volumes of two spheres are proportional to the cubes of their radii.

CHAPTER V

THE LAWS OF VARIATION

To use mathematics effectively in scientific work, we must know the meaning of certain common statements concerning the variation of quantities, and must be able to translate these statements into equations or formulas.

In this section we shall study how quantities which depend upon each other vary. For example, does an increase in the value of one varying quantity cause a corresponding increase in the related quantity? Or does an increase in one varying quantity cause a corresponding decrease in the other? How are these relations expressed by formulas?

There is a close relation between variation and proportion. Problems that can be stated in terms of one can be stated in terms of the other. This fact will become evident after a careful study of the following discussion.

A Variable is a quantity whose value is not fixed, but may be any one of a set of numbers.

A Constant is a quantity whose value does not change in the course of a discussion. For example, $A = \pi r^2$. Here A and r are variables, while π is a constant whose value is fixed, namely 3.1416(approximately). Obviously the arithmetical numbers 1, 2, $3\frac{1}{2}$ --- are constants.

Direct Variation.

If a man walks three miles an hour, the distance he travels in a number of hours is represented by the equation D = 3t. In this equation the value of D depends upon the value of t. The following table represents the relation between these variables:

If t is	1	2	3	6	8	10
Then D is	3	6	9	18	24	30

The table shows: (1) That the greater the number of hours, the greater the distance traveled. (2) That the ratio of any distance to the corresponding number of hours is 3; that is, the ratio of the two quantities is constant. Then D is said to vary directly as t. In mathematics this is expressed by $\frac{D}{t} = 3$ or $D = 3t$.

In general, if x varies directly as y, the formula is $x \propto y$, or $\frac{x}{y} = K$, or $x = Ky$, where K is the constant of variation.

The value of K can be found if we know the value of x corresponding to any value of y. Thus if x = 8 when y = 2, we have $K = \frac{8}{2} = 4$. The definite formula for x would be x = 4y.

Furthermore, if we take from the above table any two values of D such as 9 and 24 and the two corresponding values of t, which are 3 and 8, the ratio of these two values of D is equal to the ratio of the two values of t, $\frac{24}{9} = \frac{8}{3}$. This is written in the form of a proportion $\frac{D_1}{D_2} = \frac{t_1}{t_2}$. Thus D is directly proportional to t.

Note. The following expressions have the same meaning:
1. y varies directly as x,
2. y varies as x,
3. y is directly proportional to x,
4. y is proportional to x.

Does the area of a triangle vary as the base?

Does the interest vary directly as the time?

Is the circumference of a circle directly proportional to the radius?

Is the area of a circle directly proportional to the radius?

One variable may vary as the square of another. Thus the area of a circle varies, not as the radius, but as the square of the radius. $A = K r^2$ (here $K = \pi$). If the radius is doubled, the area becomes 4 times as large. If the radius becomes 3 times as long, the area becomes 9 times as great.

A variable may vary as the cube of another. The volume of a sphere varies as the cube of the radius. $V = K r^3 (K = \frac{4}{3}\pi)$.

It is proved in geometry that the areas of two similar figures are proportional to the squares of any two corresponding sides; and that the volumes of any two similar figures are proportional to the cubes of any two corresponding dimensions.

A bucket 10 inches deep will hold ? times as much water as a similar one 5 inches deep?

How much per dozen could you afford to pay for oranges 4 inches in diameter, if oranges 2 inches in diameter sell for 30 cents a dozen? (Assume both kinds of the same quality).

Joint Variation.

The area of a triangle is $A = \frac{1}{2} bh$. Thus the area varies as the base and the height. When a variable varies as the product of two or more variables, we have a case of joint variation.

The symbolic form is $x = K y z$ or $\frac{x}{yz} = K$.

Does the pay received by a workman vary jointly as his daily wage and the number of days he works?

Inverse Variation.

Some variables are so related that an increase in one is accompanied by a corresponding decrease in the other. For example, if a man allows 120 sq. ft. for a rectangular flower garden, and makes the rectangle 12 ft. long, the width must be 10 ft.; but if he makes it twice as long, it will be only half as wide, and so on.

When two variables are so related that when one of them increases, the other decreases in the same ratio, they are said to vary inversely, or to be inversely proportional. To say that x varies inversely as y means that x varies directly as the reciprocal of y. This is written $X = \frac{K}{y}$, or $xy = K$, that is, the product of the variables is constant.

One variable is inversely proportional to another when the ratio of any two values of one of them is proportional to the inverse ratio of the corresponding values of the other. The volume of a gas is inversely proportional to its pressure, may be written

$$\frac{V_1}{V_2} = \frac{P_2}{P_1}, \quad \text{or } P\,V = K, \text{ or } V = \frac{K}{P}.$$

Combined Variation.

In some examples one quantity may vary directly as another and inversely as a third. The time required to dig a ditch varies directly as the length of the ditch in feet and inversely as the number of men working on the job. That is,

$$t = \frac{K\,f}{n}.$$

Note. In problems of variation observe the following steps:

1. Express the variation as a formula.
2. Determine the constant K from the given facts. (In some problems it is not necessary to determine the value of K).
3. Rewrite the formula in its more exact(definite) form obtained by substituting for K its value.
4. Evaluate this formula for the unknown.

Exercise.

Express the following relations by means of equations:

1. The area of a parallelogram varies jointly as the base and altitude.
2. The distance a body falls from a position of rest varies as the square of the number of seconds in which it falls.

3. The interest varies jointly as the principal, the rate, and the time.

4. The weight of a body above the earth's surface varies inversely as the square of its distance from the center of the earth.

5. Express the formulas of the problems on page 5 in terms of variation.

6. If V varies jointly as h and as the sqaure of r, and V = 132 when h = 14 and r = 3, find V when h = 10, and r = 4.

7. How can you test for direct variation? For inverse variation? Are x and y directly proportional in the equation $x = 2y + 5$?

8. To double the radius of a sphere has what effect on its surface? on its volume?

9. The illumination of an object varies inversely as the square of its distance from the source of light, and the illumination of an object at the distance of 8 ft. from the source of light is 3. Find the illumination of an object at 32 ft. from the source of light; at the distance of 20 ft.

10. If a book is being held at a distance of six feet from a lamp, how much nearer must it be brought in order that the illumination on the page shall be doubled?

11. If two men 160 lb. and 190 lb. respectively are balanced on the ends of a 12 ft. pole, what is the length from the fulcrum to each end?

12. When the volume of air in a bicycle pump is 24 cu. in., the pressure on the handle is 30 lb. Later when the volume of air is 20 cu. in., the pressure is 36 pounds. Determine the law of variation, and find the volume of the air when the pressure is 42 lb.

13. The squares of the periods of the planets vary directly as the cubes of their mean distances from the sun. Find the period(length of year) of Mercury which is 36 million miles from the sun. The earth is 93 million miles from the sun.

14. The number of units of heat H generated by an electric currect of I amperes in a circuit, varies as the square of the current I, as the resistance R, and as the time t in seconds during which the current passes. Write the formula.

15. By trial in the formula of problem 14, it is found that H = 388,000 if I = 10, R = 9, and t = 30 minutes. Find H when I = 40, R = 50 and t = 45 minutes.

The Strength of a Beam.

Other things being equal, the strength of a rectangular beam supported at both ends, varies (1) directly as the breadth in inches, (2) directly as the square of the depth in inches, and (3) inversely as the length of the span in feet. The formula is

$$S = \frac{K\ b\ d^2}{L}$$

By the expression "other things being equal" is meant that the material must be the same, and the beams must be similarly supported and similarly loaded. The constant K is the same when these "other things" are equal.

To find this constant it is usual to take a bar of similar material, one inch square in cross section and long enough to allow of its being placed on supports one foot apart. The constant to be used with beams weighted in the middle is the weight of the central load which is just sufficient to break the bar.

In beams having the load uniformly distributed, the constant is twice as large as when the load is centrally located.

Forrthe man who wishes to apply these facts, another important consideration is the ratio which the safe load bears to the breaking load. This ratio is called the factor of safety. Its value for a dead load(stationary load) is usually taken as 5; for a live load(moving load) it is often taken as 10.

16. The constant for white pine is 300 lb. Find the weight that a beam of this material centrally loaded, and 10 ft. long, 3 in. broad, and 7 in. deep, will support. How much is the safe load if the factor of safety is 5?

17. How long may a beam of white pine centrally loaded, be between supports, if it is to have a safe load of 750 lbs., and it is 3 in. by 8 in. set on edge?

18. By experiment it is found that a beam of pine 40 ft. long, 1 ft. broad, and 1 ft. deep, will carry a load of 4500 lb. Find the depth of a beam of the same wood similarly loaded to carry a load of 1200 lb. When the length is 6 ft. and the breadth is 2 inches.

19. The joists in a room 14 ft. wide and 26 ft. long are 3 in. by 10 in. How far should their centers be placed apart if the floor is to support a crowd of men? K = 1100#; a crowd of men closely packed average 140 lb. per square foot). Ans. 2 ft.

Chapter VI.

P E R C E N T A G E. M I X T U R E S.

Percentage is a special form of decimal fraction in which the denominator is 100. The symbol % stands for per cent, which means hundredths, or by the hundred. Thus,

6% means $\frac{6}{100}$ or .06

225% means $\frac{225}{100}$ or 2.25

.3% means $\frac{.3}{100}$ or .003

Percentage is especially adapted to business and to comparison of data in science.

Example. An analysis of one pound of alfalfa showed that $\frac{1}{4}$ of a pound was weed seed, 1/5 of a pound dirt, and 11/20 of a pound good seed.

The fractions $\frac{1}{4}$, 1/5 and 11/20 cannot be easily compared to show the relative amounts of the ingredients. If we reduce them to the same denominator 100, the analysis shows:

Weed seed $\frac{1}{4}$ = .25 or 25%

Dirt $\frac{1}{5}$ = .20 or 20%

Good seed $\frac{11}{20}$ = .55 or 55%

There are three kinds of problems in percentage.

Problem (1). To find a given per cent of a number, as 8% of 25.

Problem (2). To find what per cent one number is of another; as what per cent is 6 of 15?

Problem (3). To find a number when a certain per cent of it is given; as 5 is 8% of what number?

Problems in percentage may often be more easily solved by algebra than by arithmetic. The percentage formula is

Percentage = rate × base or

$P = r\,b.$

This formula contains three letters. If any two are given, the third may be found.

Exercise.

1. What is 4% of 60?
2. What is 5% of x?
3. What is 6% of (x + 20)?
4. 8 is $3\frac{1}{2}$% of what number?

5. 6% of x is 20. Find x.

6. Find x if x-5% of x = 220.

7. What per cent of 25 is 6?

8. x + 8¼% of x = 360. Find x.

9. What is the cash cost of a plow at $155 subject to a 12% discount and a further discount of 3% for cash?

10. A farmer sold his milk to a dairy company and received credit for 75 pounds of butter fat. If his milk tested 3.6% butter fat, how many pounds of milk did he sell?

11. A bankrupt firm having assets of $22,500 and liabilities of $35,600 will pay what per cent of its indebtedness?

12. If 16% by weight of wheat is lost in grinding it into flour, how many pounds of wheat are needed to produce 460 pounds of flour?

13. Ear corn shrinks in weight in the crib on an average of 3% a month. What would be the loss from shrinkage on 1250 bushels of corn if kept from Nov. 1 to Feb. 1? Which would be the more profitable, to sell the green corn in November at 52 cents or to keep it until February, and sell it at 64 cents per bushel?

1.4 A certain fertilizer for corn land contains 45% cotton-seed meal, 38% phosphorus, and 17% potash. How many pounds of each are there in a ton of fertilizer?
How much acid phosphate containing 18% phosphorus is required to supply the phosphorus in a ton of fertilizer?

15. What per cent above cost must a merchant mark his goods so as to allow a discount of 15% and still make a profit of 20% on the cost?

16. The experience of a grocer shows that 10% of the sales are bad debts. At what price must he sell an article costing $5 so as to make a profit of 25% on the cost and allow for the bad debt factor?

Mixtures.

It is often necessary for a merchant or a farmer to know the results of mixing two varieties of the same substance which differs in the percentage of some ingredient; such as two grades of milk, vinegar, medicine, etc.

Example 1. A farmer receives an order for 120 pounds of 4% milk. How many pounds must he take of each of two grades containing 5% and 3½% butter fat in order to fill the order?

Solution: Let x = number of pounds of 5% grade.
Then, 120 - x = number of pounds of 3½% grade.
5% of x + 3½% of (120-x) = 4% of 120.
$.05x + .035(120-x) = .04 \times 120.$
Multiplying each side by 1000,
$50x + 4200 - 35x = 4800$
$15x = 600$
$x = 40$ lbs. of 5% grade
$120-x = 80$ lbs. of 3½% grade.

Example 2. How many pounds of 5.4% milk must be added to 50 pounds of 3.8% milk to produce a 4.5% milk?

Solution: Let x = the number of pounds of 5.4% milk.
Then, 50 + x = the number of pounds of 4.5% milk.
.
... 5.4% of x + 3.8% of 50 = 4.5% of (50 + x)
or $.054x + .038 \times 50 = .045(50+x)$
$54x + 1900 = 2250 + 45x$
$9x = 350$
$x = 38.9$ lbs. of 5.4% milk.

The Pearson Rule is a graphic method of solving mixture problems of this kind.

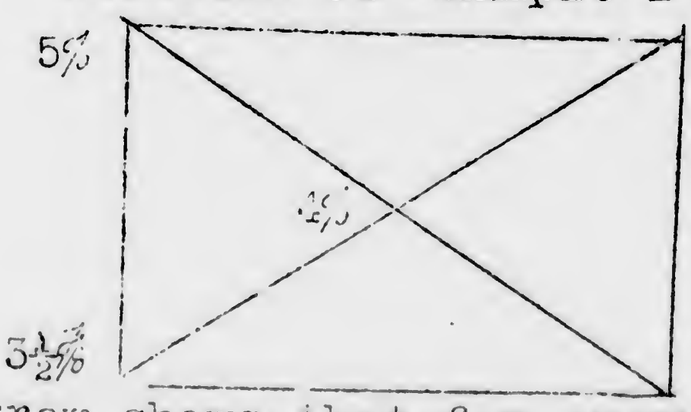
(1) Draw a rectangle with two diagonals, and in the center place the percentage desired (4%).

(2) Place the percentages of the substances to be mixed at the left hand corners (5% and 3½%).

(3) Subtract diagonally and place the remainders at the right hand corners (½ and 1).

(4) Each remainder indicates the amount to be taken from the substance whose percentage is in a horizontal line with that remainder.

Solution of Example 1 by the Pearson rule:



½ ------ x lbs.

1 ------- 120-x lbs.

The diagram shows that for every ½ pound of 5% milk, one pound of 3½% milk should be taken.

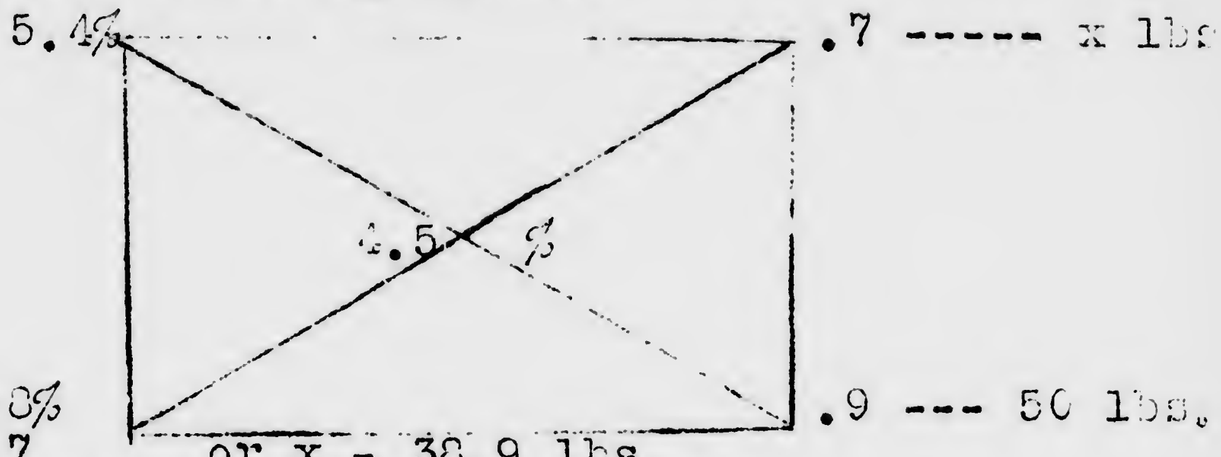
$$\frac{x}{120-x} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$2x = 120 - x$

$x =$ 40 lbs. of 5% grade.

$120 - x =$ 80 lbs. of 3½% grade.

Solution of Example 2.



$$\frac{x}{50} = \frac{.7}{.9}.$$ or x = 38.9 lbs.

Exercise.

1. How many pounds each of 3.5% milk and 6.2% milk are required to make 100 pounds of 4.5% milk?

2. How many pounds each of 25% cream and 3% milk are required to make 5% milk? (Consider an arbitrary quantity of the mixture, say 100 pounds).

3. How many pounds of skim milk must be added to 100 pounds of 6% milk to form a standardized milk containing 4.5% butter fat?

4. How much water must be added to 20 gallons of alcohol 90% pure to reduce it to a 60% solution? (Consider a mixture of 100 pounds).

5. What kind of milk is a mixture of 10 pounds of 4.5% milk and 15 pounds of 5.2% milk?

6. What are the proportions of 5.1% milk and 3.4% milk required to obtain a 4% milk?

7. How much skim milk must be mixed with a 5.6% milk to make it 4% milk?

8. How much water must be added to a barrel of vinegar 85% pure, to reduce it to a 45% solution?

9. What per cent of evaporation must take place from a 6% solution of salt and water (6% by weight is salt) to make the remaining portion an 8% solution?

10. A dealer has two grades of coffee costing 32 cents and 40 cents per pound. He wishes to mix them so that 75 pounds of the mixture will be worth 35 cents per pound. How many pounds of each kind must be used in the mixture?

11. An alloy of copper and silver weighs 40 pounds and contains 5 pounds of copper. How much silver must be added so that the new alloy will contain 5% copper?

Chapter VII.

E X P O N E N T S

The fundamental definition when a is a positive integer, is:

$x^a = x \cdot x \cdot x \cdot x$ ---- to a factors.

There are five laws of exponents.

1. The multiplication law.

$$x^a \cdot x^b = x^{a+b}$$

$$3^2 \cdot 3^4 = 3^6 \; ; \; 2^3 \cdot 2^2 \cdot 2^4 = 2^9.$$

To multiply two powers of the same base, add the exponents.

II. The division law.

$$x^a \div x^b = x^{a-b} \; ; \; 5^6 \div 5^2 = 5^4.$$

To divide one power by another of the same base, subtract the second exponent from the first.

III. Power of a Power.

$$(x^a)^b = x^{ab}; \; (4^2)^3 = 4^6.$$

To find the power of a power multiply the exponents.

IV. Power of a Product.

$$(xy)^a = x^a \cdot y^a \; ; \; (3 \times 2)^4 = 3^4 \times 2^4 .$$

The power of a product is the same as the product of the powers of the factors.

V. Power of a Fraction.

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \; ; \quad \left(\frac{3}{4}\right)^5 = \frac{3^5}{4^5} .$$

The power of a fraction equals the power of the numerator divided by the power of the denominator.

If we adopt the above laws as the universal laws for all rational exponents, whether positive or negative, integral, fractional, or literal, we can find a meaning for zero exponent, negative exponent, and fractional exponent.

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If we adopt the above laws as the universal laws for all rational exponents, whether positive or negative, integral, fractional, or literal, we can find a meaning for zero exponent, negative exponent, and fractional exponent.

Meaning of Zero Exponent.

What meaning must be given to x^0 ?

$$x^3 \div x^3 = x^0 \quad \text{and} \quad x^3 \div x^3 = 1$$

$$\therefore x^0 = 1; \quad 12^0 = 1 ; \quad (-4a)^0 = 1$$

Definition. Any quantity (not zero) with a zero exponent equals one.

Meaning of Negative Exponent.

What meaning must be given to x^{-a} and x^{-3}?

$$x^{-a} = \frac{x^{-a}}{1} \cdot \frac{x^a}{x^a} = \frac{x^{-a+a}}{x^a} = \frac{x^0}{x^a} = \frac{1}{x^a}.$$

Likewise, $x^{-3} = \frac{1}{x^3}$; $\frac{1}{2x^{-4}} = \frac{x^4}{2}$; $a5^{-2} = \frac{a}{25}$.

Definition. A negative power of a quantity is equal to 1 divided by the corresponding positive power.

Meaning of a Fractional Exponent.

If the law that $(x^a)^b = x^{ab}$ is to hold for fractional exponents, we would have

$x^{2/3} \cdot x^{2/3} \cdot x^{2/3} = x^{6/3} = x^2$. This means that $x^{2/3}$ is the cube root of x^2, that is, $x^{2/3} = \sqrt[3]{x^2}$. Again, $\left(x^{a/b}\right)^b = x^a$, or $x^{a/b} = \sqrt[b]{x^a}$. From this it follows that $x^{\frac{1}{2}}$ means the square root of x; that is, $x^{\frac{1}{2}} = \sqrt{x}$. Also $x^{1/3} = \sqrt[3]{x}$.

Definition. In a fractional exponent, the numerator indicates a power and the denominator indictaes a root. That is, $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ or $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$.

Note. In handling exponents, indices, and radicals, it often simplifies the work to use the fractional exponents.

Scientific Notation. Exponents Used in Writing Numbers.

Often in scientific subjects we see such expressions as

9.29×10^7 or 2.32×10^{-5}. This form is shorter than the common notation and furnishes an easier way of comparing numbers which have many zeros. When the decimal point is placed after the first significant figure the number is said to be in its standard form.

Distance to sun, $92{,}900{,}000 = 9.29 \times 10{,}000{,}000 = 9.29 \times 10^{7}$.

Length of light wave, $2.32 \times 10^{-5} = 2.32 \times \frac{1}{100{,}000} = .0000232$.

Exercise.

Give in simplest form the values of the following:

1. $3^{0} \cdot 8^{1/3} \cdot 2^{-3}$.
2. $1^{-4} + 1^{4} + 1^{0} + 1^{1}$.
3. $4^{-4} \cdot 4^{\frac{1}{2}} \cdot 4^{-2}$.
4. $(-x)^{-2} \cdot (-y)^{3}$
5. $27^{4/3} \cdot 32^{-2/5}$.
6. $\frac{1}{x^{-2}} + \frac{1}{y^{-2}}$
7. $2x^{-a} - \frac{1}{2x^{-a}}$.
8. $x + \frac{1}{2}y^{-3}$.
9. $(x - y)^{-2}(x^{2} - y^{2})$.
10. $(a^{-2} + b^{3})^{0}$.
11. $10^{3.125} \times 10^{-2} \times 10^{0}$.
12. $\frac{a^{3} b^{2} c^{-4}}{a^{-1} b^{-2} c}$.
13. $\frac{x^{3}}{x^{-2} + x^{-4}}$. Multiply Num. and Den. by x^{4}.
14. $\frac{2x^{-\frac{1}{2}} + y^{-\frac{1}{2}}}{x}$. Multiply Num. and Den. by $x^{\frac{1}{2}} y^{\frac{1}{2}}$.
15. $(\frac{3}{4})^{-2} \cdot \frac{125}{27}^{-1/3}$.
16. $(x^{-\frac{1}{2}} y^{2/3})^{4}$.
17. $\left(\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}\right)^{2/3}$.
18. $(x^{a+1} - y^{a-1})^{2}$.
19. $\frac{x - x^{-1}}{x^{1/3} - x^{-1/3}}$.
20. $(e^{x} + e^{-x})(e^{x} + e^{-x})$.

Express the following in the common notation:

21. 3.76×10^{9}.
22. 4×10^{-7}.
23. 2.798×10^{-6}.
24. 756.0×10^{12}.

Express the following in the standard notation:

25. 367,000,000,000 .
26. 0.000,000,523.
27. 467,300,000 .
28. 0.000,000,004,786.

Chapter VIII.

R A D I C A L S

A radical is a root of a number indicated by the radical sign, as $\sqrt{3}$, $\sqrt[3]{x}$, $\sqrt[4]{a-1}$.

The number underneath the radical sign is called the radicand. The index determines the order of the radical. If the indicated root can be obtained exactly, the radical is a rational number; as $\sqrt{9}$, $\sqrt[4]{16}$, $\sqrt[3]{-27}$. If it cannot be obtained, the radical is an irrational number; as, $\sqrt{2}$, $\sqrt[3]{5}$.

Three principles are used frequently in simplifying radicals.

I. $\sqrt[n]{x \cdot y} = \sqrt[n]{x} \cdot \sqrt[n]{y}$. Thus $\sqrt{5.9} = \sqrt{5} \cdot \sqrt{9}$.

II. $\left(\sqrt[n]{x}\right)^n = x^{n/n} = x$, Thus $\left(\sqrt{2}\right)^2 = \sqrt{2^2} = 2$.

III. $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$. Thus $\sqrt{\frac{5}{7}} = \frac{\sqrt{5}}{\sqrt{7}}$.

A radical may often be simplified by factoring. Thus,

$$\sqrt{300} = \sqrt{4 \cdot 25 \cdot 3} = \sqrt{4} \cdot \sqrt{25} \cdot \sqrt{3} = 2 \cdot 5\sqrt{3} = 10\sqrt{3}.$$

Sometimes it is desirable to inverse the process and introduce a factor under the radical sign. Thus,

$$5\sqrt{2} = \sqrt{5^2} \cdot \sqrt{2} = \sqrt{50} \; ; \; x\sqrt[3]{y} = \sqrt[3]{x^3} \cdot \sqrt[3]{y} = \sqrt[3]{x^3 y}$$

The student will save time if he will memorize

$$\sqrt{2} = 1.414 \; ; \sqrt{3} = 1.732; \sqrt{5} = 2.236; \sqrt{6} = 2.449.$$

Note. The square root of 9 is both +3 and -3, since $(+3)(+3) = 9$ and also $(-3)(-3) = 9$. The symbol $\sqrt{9}$, however, means only +3, the principal root. But $-\sqrt{9} = -3$ and $-\sqrt[4]{16} = -2$. The symbol $\pm\sqrt{9}$ denotes both +3 and -3. Further $\sqrt[3]{27} = +3$, $\sqrt[3]{-27} = -3$, while $-\sqrt[3]{-27} = +3$. An understanding of these symbols will avoid ambiguity, especially in solving radical equations.

For purposes of computation, it is often desirable that the denominator of a fraction be made rational. This can be done by multiplying the numerator and denominator by the proper factor. Thus

$$\sqrt{\frac{3}{2}} = \sqrt{\frac{3}{2} \cdot \frac{2}{2}} = \tfrac{1}{2}\sqrt{6} = \frac{2.449}{2} = 1.2245.$$

$$\sqrt[3]{\tfrac{3}{4}} = \sqrt[3]{\frac{3}{4} \cdot \frac{2}{2}} = \tfrac{1}{2}\sqrt[3]{6}.$$

The process of making the denominator under the radical sign a rational number is called rationalizing the denominator.

Exercise.

Find the value of the following correct to three decimal places:

1. $\sqrt{75}$. 2. $\sqrt{180}$. 3. $\sqrt{294}$.

4. $\sqrt{20} + \sqrt{45}$. 5. $\sqrt{\frac{2}{3}}$. 6. $\sqrt{\frac{8}{27}}$.

7. $\sqrt{808}$. 8. $\sqrt{160}$. 9. $\sqrt{3.12}$.

Simplify:

10. $\sqrt[3]{16}$. 11. $\sqrt{\frac{1}{2x}}$. 12. $\sqrt{4a\,b^2\,c^3}$.

13. $\sqrt{\frac{3}{a^3}}$. 14. $\frac{\sqrt[4]{4 + \sqrt{\frac{4}{3}}}}{\sqrt{2}}$. 15. $\frac{2+\sqrt{5}}{\sqrt{3}}$.

16. $\left(\frac{8x}{5a^3}\right)^{\frac{1}{2}}$. 17. $\sqrt{r^2 - \left(\frac{r}{4}\right)^2}$. 18. $\sqrt{81 - \left(\frac{9}{2}\right)^2}$.

19. $\sqrt{6^2 + 9^2}$. 20. $\sqrt[3]{\frac{2}{3}}$. 21. $\sqrt{a^4 + a^2 b^2}$

Introduce the coefficient under the radical:

22. $3\sqrt{2}$. 23. $2\sqrt[3]{5}$. 24. $\frac{1}{2}\sqrt{3}$.

25. $(1+a)\sqrt{1+a}$. 26. $(x+1)\sqrt{\frac{x-1}{x\ 1}}$. 27. $\frac{x}{a}\sqrt{x + a}$.

Addition and Subtraction of Radicals.

Only similar radicals, that is, radicals of the same order which have like radicands, can be added or subtracted and the result expressed by one term. Thus

$$\sqrt{75} - \sqrt{12} + 2\sqrt{48} = 5\sqrt{3} - 2\sqrt{3} + 8\sqrt{3} = 11\sqrt{3}.$$

$\sqrt{8} + \sqrt{12} = 2\sqrt{2} + 2\sqrt{3}$. These cannot be added into one term, since they are dissimilar.

Multiplication and Division of Radicals.

Only radicals of the same order can be multiplied or divided and the result expressed under one radical. Thus

$$\sqrt{5} \cdot \sqrt{3} = \sqrt{15}\ ;\ \sqrt[3]{x^2} \cdot \sqrt[3]{x^4} = \sqrt[3]{x^6} = x^2.$$

$$\sqrt{10} \div \sqrt{2} = \sqrt{5}\ ;\ \sqrt{3a^2} \div \sqrt{ax^2} = \sqrt{\frac{3a^2}{ax^2}} = \frac{1}{x}\sqrt{3a}.$$

If the radicals are not of the same order they can be reduced to the same order by expressing the radicals with fractional exponents and reducing the exponents to a common denominator.

$$\sqrt[3]{x} \cdot \sqrt{x} = x^{1/3} \cdot x^{\frac{1}{2}} = x^{2/6} \cdot x^{3/6} = x^{5/6} = \sqrt[6]{x^5}.$$

Exercise.

Perform the indicated operations and simplify:

1. $\sqrt{45} - \sqrt{20} + \sqrt{80}$.

2. $\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{4}} + \sqrt{\frac{7}{8}}$.

3. $\sqrt[3]{135} - \sqrt[3]{320}$.

4. $\sqrt{60} - \sqrt{15} + 2\sqrt{\frac{3}{5}}$.

5. $\sqrt{150} - 3\sqrt{24}$.

6. $\sqrt[3]{27x^4} + \sqrt[3]{24x}$.

7. $2\sqrt{3} \cdot 3\sqrt{2}$.

8. $\sqrt[3]{16} \cdot \sqrt[3]{4}$.

9. $\sqrt[3]{2a^2} \cdot \sqrt{8a}$.

10. $(\sqrt{2} + 3)(3\sqrt{2} - 5)$.

11. $(R + \sqrt{2})(R - \sqrt{2})$.

12. $(2\sqrt{3} - 1)^2$.

13. $(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})$.

14. $(2a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$.

15. $(3\sqrt{x - 2})^2$.

16. $\sqrt[3]{20} \div \sqrt[6]{125}$.

17. $\sqrt[3]{\frac{4}{9}} \div \sqrt{\frac{2}{3}}$.

18. $\sqrt[3]{2x^2y} \div \sqrt{2xy}$.

19. $\dfrac{2}{2 - \sqrt{3}} = \dfrac{2}{2 - \sqrt{3}} \cdot \dfrac{2 + \sqrt{3}}{2 + \sqrt{3}} = \dfrac{4 + 2\sqrt{3}}{4 - 3} = 4 + 2\sqrt{3}$

$= 4 + 3.464 = 7.464.$

Conjugates.

Since $(a - \sqrt{b})(a + \sqrt{b}) = a^2 - b$, either of these factors can be used to rationalize the other. Such two binomilas are called conjugate radicals.

20. $8 \div (3 + \sqrt{6})$.

21. $9 \div (3 - \sqrt{2})$.

22. $5 \div (\sqrt{5} - \sqrt{2})$.

23. $\dfrac{3\sqrt{2}}{\sqrt{3} + \sqrt{2}}$.

24. $\dfrac{2\sqrt{3} - 1}{3\sqrt{2} + 4}$.

25. $\dfrac{2\sqrt{x} + y}{3\sqrt{x} - 2y}$.

Imaginary Numbers.

In the equation $x^2 + 4 = 0$ or $x^2 = -4$, if we take the square root of both sides we have $x = \pm\sqrt{-4}$. Now, what does $\sqrt{-4}$ mean? Is -2 the square root of -4? No, for $(-2)^2 = +4$. Is $+2$? No, for $(+2)^2 = +4$. Since the square of any real number, positive or negative, is a positive number, there can be no real number whose square is -4. Consequently, the square root of a negative number cannot be a real number.

This has led to the extension of the number concept. Numbers whose squares are negative real numbers are called imaginary numbers. And conversely, the square roots of negative numbers are imaginary numbers.

As the unit of the real number system is 1, so the unit of the system of imaginary numbers is $\sqrt{-1}$. This number occurs so frequently that a special symbol is used for it; that is, $\sqrt{-1} = i$. Furthermore, any imaginary number can be written as the product of a real number and $\sqrt{-1}$. Thus,

$$\sqrt{-4} = \sqrt{4(-1)} = 2\sqrt{-1} = 2i.$$

$$\sqrt{-5} = \sqrt{5}\sqrt{-1} = i\sqrt{5}.$$

It must be clearly understood that while the product of $\sqrt{2} \cdot \sqrt{3} = \sqrt{2 \cdot 3} = \sqrt{6}$; the product of $\sqrt{-2} \cdot \sqrt{-3}$ does not equal $\sqrt{(-2)(-3)}$.

The product of imaginaries should first be written

$$(\sqrt{2}\sqrt{-1}) \cdot (\sqrt{3}\sqrt{-1}) = \sqrt{6}\,(\sqrt{-1})^2 = -\sqrt{6}.$$

Since $i = \sqrt{-1}$, $i^2 = -1$; $i^3 = i^2 \cdot i = -i$; $i^4 = i^2 \cdot i^2 = +1$.

We operate with i as with any other letter.

The sum of a real number and an imaginary number is called a complex number as $2 - 3i$, $-1 + 5i$, $a + bi$.

Exercise.

Simplify:

1. $\sqrt{-9} + \sqrt{-4}$.
2. $\sqrt{-1} - \sqrt{-25} + \sqrt{-49}$.
3. $\sqrt{5} \cdot \sqrt{-20}$.
4. $\sqrt{-a^2} - \sqrt{4a^2}$.
5. $\sqrt{-x^2 - 2x - 1}$.
6. $\frac{3c}{2} + \sqrt{-\frac{27}{4}}$.
7. $(4 + \sqrt{-5})(2 - \sqrt{-6})$.
8. $(\sqrt{-3})^2 - (2\sqrt{-4})^2$.
9. $\sqrt{-6} \div \sqrt{-2}$.
10. $5 \div (2\sqrt{-5})$.

Chapter IX.

Q U A D R A T I C E Q U A T I O N S

Linear Equations.

An equation of the first degree, that is, an equation involving only the first power of the unknown quantity, is called a linear equation. The student is already familiar with the method of solving linear equations with one unknown.

Quadratic Equations.

An equation in which the highest power of the unknown is the second power, is called an equation of the second degree, or a quadratic equation. Many problems in science and mechanics are solved by quadratic equations. Furthermore, a thorough understanding of the quadratic equation is necessary for the study of analytic geometry and some parts of trigonometry. For these reasons the student should master the methods of solving quadratic equations.

The Standard Form.

Any quadratic equation in one unknown can be written in the form

$$Ax^2 + Bx + C = 0$$

Where A, B, C do not involve x, and where $A \neq 0$.

Examples: (1) $3x^2 - 2x + 5 = 0.$

(2) $5x^2 + (m - 3)x - 4 = 0.$

Solution by Factoring.

Some quadratic equations can be easily solved by factoring. The solution is based upon the following

Principle. If the product of two factors is zero, one of the factors must be zero.

Example, $6x^2 + x = 15.$

Transposing, $6x^2 + x - 15 = 0.$

Factoring, $(3x + 5)(2x - 3) = 0.$

Now we have a product equal to zero. Therefore, one of its factors must be zero.

If $3x + 5 = 0$, $3x = -5$ or $x_1 = -\frac{5}{3}$

If $2x - 3 = 0$, $2x = 3$ or $x_2 = \frac{3}{2}$.

Check: $\frac{50}{3} - \frac{5}{3} = 15$. $\therefore x_1 = -\frac{5}{3}$ is a root.

$\frac{27}{2} + \frac{3}{2} = 15$. $\therefore x_2 = \frac{3}{2}$ is a root.

Note. Every equation of the second degree has two roots. In some cases the roots are equal.

Method of Solving an Equation by Factoring.

(1) Transpose all terms to the left member making the right member zero.

(2) Factor the left member completely.

(3) Set each factor equal to zero.

(4) Solve the resulting equations.

(5) Check the results obtained in the original equation.

Exercise.

Solve by factoring:

1. $x^2 - x - 12 = 0$.
2. $x^2 + 5x + 4 = 0$.
3. $2x^2 - x = 36$.
4. $\frac{1}{2}x^2 + x = 12$.
5. $2x^2 - 5x = 0$.
6. $(x^2 - 1) + (x - 1)^2 = 0$.
7. $x^2 - a^2 + x + a = 0$.
8. $x^2 - 2x + 5x - 10 = 0$.
9. $6x^2 + 7x = 0$.
10. $y^2 - 7y - 8 = y + 1$.
11. $(x - 3)^2 - 2(x - 3) = 8$.
12. $(x - a)^2 - b^2 = 0$.

Solution of a Quadratic by Completing the Square.

Any quadratic equation can be solved by completing the square, that is, by making the left side of the equation a perfect square. This method is used when the quadratic expression cannot be easily factored.

To understand this method observe that, in the square of any binomial such as

$$(x - 5)^2 = x^2 - 10x + 25,$$

$$\text{or,} \quad (x + a)^2 = x^2 + 2ax + a^2,$$

the final terms is the square of one-half the coefficient of x.

Example, $2x^2 + 5x - 7 = 0.$

Transposing, $2x^2 + 5x = 7$

Dividing by 2, to make x^2 have 1 for its coefficient

$$x^2 + \frac{5}{2}x = \frac{7}{2}$$

Adding to both sides the square of one half the coefficient of x, or

$$\left(\frac{5}{4}\right)^2,$$

$$x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = \frac{7}{2} + \frac{25}{16} = \frac{81}{16}$$

The left member is now a perfect square. Extracting the square root of each side,

$$x + \frac{5}{4} = \pm \frac{9}{4}$$

$$x_1 = -\frac{5}{4} + \frac{9}{4} = 1.$$

$$x_2 = -\frac{5}{4} - \frac{9}{4} = -\frac{7}{2}$$

Check: The student should check each result.

To Solve by Completing the Square.

(1) Reduce the equation to the forn $Ax^2 + Bx = -C$

(2) Divide by the coefficient of x^2.

(3) Add to both sides the square of half the new coefficient of x, to make the left member a perfect square.

(4) Take the square root of both sides using the double sign.

(5) Solve these results for the unknown, and check.

Solve by completing the square:

1. $x^2 + 6x - 27 = 0$.
2. $3x^2 - 2x = 40$.
3. $4r^2 - 3r = 3$.
4. $4x^2 - 4x = 7$.
5. $5x^2 + x = \frac{7}{4}$.
6. $3t^2 + 5t - 8 = 0$.
7. $\frac{x^2}{2} + \frac{3x}{5} - \frac{31}{2} = 0$.
8. $\frac{x+11}{x+3} = \frac{2x+1}{x+5}$.
9. $3y^2 + 7y + c = 0$.
10. $2x + \frac{1}{x} = 3$.
11. $ax^2 + bx + c = 0$.
12. $Ax^2 + Bx + C = 0$.

Solution of a Quadratic Equation by Formula.

All quadratic equations having one unknown may be written in the form

$$Ax^2 + Bx + C = 0.$$

By the method of completing the square the roots of this equation are found to be

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

This result can be used as a formula to find the roots of any quadratic equation. In using this formula, the signs must be considered a part of the values of A, B, and C. It must be remembered also that A is the coefficient of the second degree term, B the coefficient of the first degree term, and C the constant.

Example, $3x^2 - x = 5$

Transposing, $3x^2 - x - 5 = 0$.

$A = 3, \quad B = -1, \quad C = -5.$

Substituting, $x = \frac{1 \pm \sqrt{1+60}}{6}$

$$\therefore \quad x_1 = \frac{1+\sqrt{61}}{3}; \quad x_2 = \frac{1-\sqrt{61}}{6}$$

Check: The check by substitution is a long process.

A shorter check is $x_1 + x_2 = -\frac{B}{A}$, and $x_1 \cdot x_2 = \frac{C}{A}$

$x_1 + x_2 = \frac{1+\sqrt{61}}{6} + \frac{1-\sqrt{61}}{6} = \frac{2}{6} = \frac{1}{3}$ which is $= -\frac{B}{A}$

Exercise.

Solve by formula.

1. $6x^2 - 13x + 6 = 0.$
2. $5r^2 - 5r = 21.$
3. $x^2 + 6x - 1 = 0.$
4. $7 + 2y = y^2$
5. $4x^2 - 12x + 9 = 0.$
6. $5y^2 + 7y + 3 = 0.$
7. $16t^2 - 800t + h = 0.$
8. $.3t^2 - 40t - 30 = 0.$
9. $2x^2 - (a+1)x - 3 = 0.$
10. $5x^2 + 3x - a + 4 = 0.$

Forming Equations with Given Roots.

If the roots of an equation are given the equation may be found. For if r is a root, then $(x - r)$ is a factor, and the equation may be written $(x - r_1)(x - r_2) = 0.$

For example, if the roots are 2 and $-\frac{3}{2}$, the equation is

$(x - 2)(x + \frac{3}{2}) = 0$; that is $x^2 - \frac{1}{2}x - 3 = 0.$

Exercise.

Form equations whose roots are:

1. $-3, 4.$
2. $\frac{1}{2}, -\frac{2}{3}.$
3. $\sqrt{5}, -\sqrt{5}.$
4. $\frac{1}{a}, \frac{1+a}{2}, 0.$
5. $3+\sqrt{-2}, 3-\sqrt{-2}.$
6. $1+\sqrt{3}, 1-\sqrt{3}.$

Solve the following equations and check:

7. $\sqrt{x-2} = 3.$
8. $\sqrt{x^2 - 4} = 4 - x.$
9. $y - 5\sqrt{y} + 6 = 0.$
10. $\sqrt{x+15} + \sqrt{x} = 15.$

Chapter X

Systems of Equations.

One Linear Equation in Two Variables.

We have seen that an equation of the first degree with only one variable has only one solution. For example, $2x + 3 = 11$, is satisfied by one value only; namely, $x = 4$. Any other number does not satisfy the equation.

On the other hand, consider a linear equation which has two variables, such as

$$2x + 3y = 4.$$

It is evident that many pairs of values satisfy this equation. Thus $x = 0$; $y = \frac{4}{3}$; $x = 1$, $y = \frac{2}{3}$; $x = 5$, $y = -2$; $x = -1$, $y = 2$; there is an unlimited number of pairs of values that satisfy the equation.

Solution of a Pair of Equations.

It is a common problem in mathematics to have to find one set of values which will satisfy each of two equations having two variables. For example, what pair of values will satisfy

$$2x + 3y = 4,$$

and satisfy also

$$3x - 4y = 23.$$

We find that the pair $x = 5$, $y = -2$ is the only pair that satisfies both equations.

Equations with more than one variable which have a common solution are called Simultaneous Equations.

Equations with more than one variable which have no common solution are called inconsistent, or contradictory. For example,

$$2x + 3y = 4$$

$$6x + 9y = 8$$

Solution of Simultaneous Equations.

In general, to solve a system of equations it is necessary to reduce the two equations in two unknowns to one equation in one unknown. This process is called Elimination, or getting rid of an unknown.

Case I.

When the two equations are of the first degree.
First Method. Elimination by Addition or Subtraction.

Example. $2x + 3y = 4$ (1)

$3x - 4y = 23.$ (2)

M. (1) by 4, $8x + 12y = 16$ (3)

M. (2) by 3, $9x - 12y = 69$ (4)

(3) (4), $17x = 85$ (5)

(5) 17, $x = 5$ (6)

Substituting in (1) from (6), $10 + 3y = 4$ (7)

$y = -2$

Check: Substitute $x = 5$ and $y = -2$ in (1) and (2)
Does $10 - 6 = 4$? Yes
Does $15 + 8 = 23$? Yes

To use this method we multiply each member of the first equation by a number, and each member of the second equation by another number so that the coefficients of the same variable in the resulting equations may become numerically equal. Then, add or subtract to eliminate this variable, and solve the obtained equation.

Second Method. Elimination by Substitution.

Example. $x + 2y = 6$ (1)

$2x + 3y = 11$ (2)

Solve (1) for x, $x = 6 - 2y$ (3)

Subs. (3) in (2) $2(6 - 2y) + 3y = 11$ (4)

$12 - 4y + 3y = 11$ (5)

$-y = -1$ (6)

$y = 1$ (7)

Subs. 1 for y in (3) $x = 6 - 2 = 4$ (8)

Check: Substituting 4 for x and 1 for y in (1) and (2).

Elimination by substitution consists in solving one equation for one unknown in terms of the other, and substituting this result in the other equation. This gives an equation in one unknown.

Exercise.

Solve the following:

1. $x + 3y = 14$
   $6y - 3x = 1$

2. $2x - 3y = 7$
   $5x + 4y = 29.$

3. $5r = 3s + 27$
   $3r + 2s = 20$

4. $2a + b = 14$
   $5a - 2b = 11$

5. $3y = 2x + 10$
   $3x + 7y = 31.$

6. $3x - 6y = 21$
   $2y - x + 6 = 0.$

7. $2y = 7x - 5$
   $8y - 28x = 3$

8. $2x + 4y = 9.$
   $4x + 8y = 18$

9. $\frac{y}{5} + \frac{x}{2} = 4$
   $\frac{y}{3} - \frac{x}{2} = -12$

10. $\frac{11y - 5x}{11} = \frac{3y + x}{16}$
    $8y - 5x = 1$

11. $ax - by = 1$
    $x + y = c$

12. $.4x + .3y = .8$
    $.6x + .2y = 1.5$

13. $\frac{3}{x} - \frac{5}{y} = 3$
    $\frac{5}{x} + \frac{5}{y} = 1$

14. $\frac{4}{y} - \frac{3}{x} = 2$
    $\frac{10}{y} - \frac{9}{x} = 3$

15. $\frac{x + y}{3} - \frac{25}{6} = \frac{x - y}{2}.$
    $\frac{x + y - 9}{2} = \frac{y - x - 6}{3}$

16. $5A + 3B = 7$
    $2A - 4B = -3.$

Equations of the First Degree in Three Variables.

In solving a set of three equations in three unknowns, the aim is to get from the three equations, two equations in two unknowns. This is done by eliminating the same unknown from two different pairs of equations. The two resulting equations are then solved as we have seen.

Case II. System of Equations, One Linear and One Quadratic.

Systems of this type are solved by substitution. Solve the linear equation for one unknown in terms of the other, and substitute the result in the quadratic equation. (Solve for the letter which gives the simpler expression to substitute).

Example. $x + 2y = 7$ (1)

$xy + y^2 = 12$ (2)

It is better to solve the first equation for x,

$x = 7 - 2y$ (3)

Substitute $7 - 2y$ for x in (2)

$y(7 - 2y) + y^2 = 12$

$y^2 - 7y + 12 = 0$

$(y - 3)(y - 4) = 0$

$y = 3, \quad y = 4$

By suostituting $y = 3$ in (3) we obtain $x = 1$.

Therefore, $\left.\begin{matrix} x = 1 \\ y = 3 \end{matrix}\right\}$ is a solution.

By substituting $y = 4$ in (3) we obtain $x = -1$.

Therefore, $\left.\begin{matrix} x = -1 \\ y = 4 \end{matrix}\right\}$ is a solution.

The values of x and y must be carefully paired so that each pair satisfies the original equations.

Check: Each pair satisfies equations (1) and (2).

Exercise.

Solve and group correctly the roots of the following systems:

1. $y^2 = 16x.$
   $y - x = 0.$

2. $x^2 + y^2 = 4$
   $y - 3x = 2$

3. $x^2 + xy = 12$
   $2x - 3y = 11.$

4. $2y = x - 1$
   $x^2 - xy + y = 19.$

Chapter XI.

G R A P H I C M E T H O D S

Statistics.

In our every day life we are often called upon to represent and compare quantities and numbers and to express the relationship between them. For example, the size of crops, the population of cities, the sales and profits of an industry, the increase or decrease of crime and disease, rates of insurance, conditions of labor, the problems of education, the records in athletics, and so on. When we have a large number of similar facts, or data, represented by numbers, we speak of them as statistics.

There are three methods of representing quantities and expressing the relationship between them:

1. The tabular method in which the facts are presented in a table.

2. The graphic method in which a drawing or picture is made to represent the facts.

3. The equation, or formula, which expresses the law of relationship.

This chapter aims to explain these methods.

The Tabular Method.

A table is an arrangement of numbers in columns and rows to present information and facilitate comparison. At the head of the columns there are explanatory phrases called captions, and at the edge of the rows such phrases are called stubs. Every table should have a title written where it can be most easily seen, preferably above the table. The ideal title is concise, catchy, and adequate.

The primary principle in the arrangement of a table is that figures to be compared should be brought as close together as possible. The most effective arrangement is, first, adjacent in the same column; second, adjacent in the same row. The most prominent place in the table is the first column or the first row.

Sometimes the table is made clearer if the facts are arranged in the order of size or rank. When the statistics are taken at stated intervals of time, either the stubs or the captions should be arranged in the natural order of time.

Compare the two ways of tabulating the lengths of some of the longest rivers given to the nearest hundred miles.

The Longest Rivers.

River	Length	River	Length
Amazon	3900	Missouri- Mississippi	4200
Congo	2900	Nile	4000
Danube	1800	Amazon	3900
Missouri-Mississippi	4200	Yangtse	3100
Nile	4000	Congo	2900
Rio Grande	3100	St. Lawrence	2200
St. Lawrence	2200	Danube	1800
Yangtse	3100	Rio Grande	1800

Exercise.

1. The expenses of the departments of a city one year were: Education $30,000; executive $2,000; fire $19,000; health $8,000; judicial $6,500;legislative $19,000; police $15,000; public works $35,000. Arrange these facts in a table.

2. The following figures represent the earnings in a week of 20 employees in a factory. Arrange them in a table that will show clearly the range covered and the general distribution. $27.50, $25.75, $30.00, 26.25, 27.75, 26.00, 28.50, 29.00, 26.50, 24.50, 31.25, 28.10, 31.10, 27.60, 26.40, 29.75, 32.20, 31.75, 29.25, 24.80.

3. The number of employees in an automobile factory was:

1920, 200 union and 400 non-union.
1921, 300 union and 500 non-union.
1922, 500 union and 600 non-union.
1923, 600 union and 500 non-union.
1924, 800 union and 400 non-union.
1925, 1200 union and 500 non-union.

Express these fates in tabular form.

4. Make a table of statistics relating to the growth of your college during the last ten years showing the attendance at the various schools.

5. Make a table showing the maximum and minimum temperatures at State College for the different days of last week.

The Statistical Graph.

The relations between numbers in a long table of statistics may often be made more vivid by a picture or diagram which appeals to the eye and enables the reader to see the relation between the quantities represented. These pictures are called graphs. They are very common in newspapers, magazines, textbooks, etc. It is important for an intelligent person to learn how to interpret and how to construct a graph.

Many kinds of graphs are used but only some of the chief types may be explained here.

I. Maps or Cartograms.

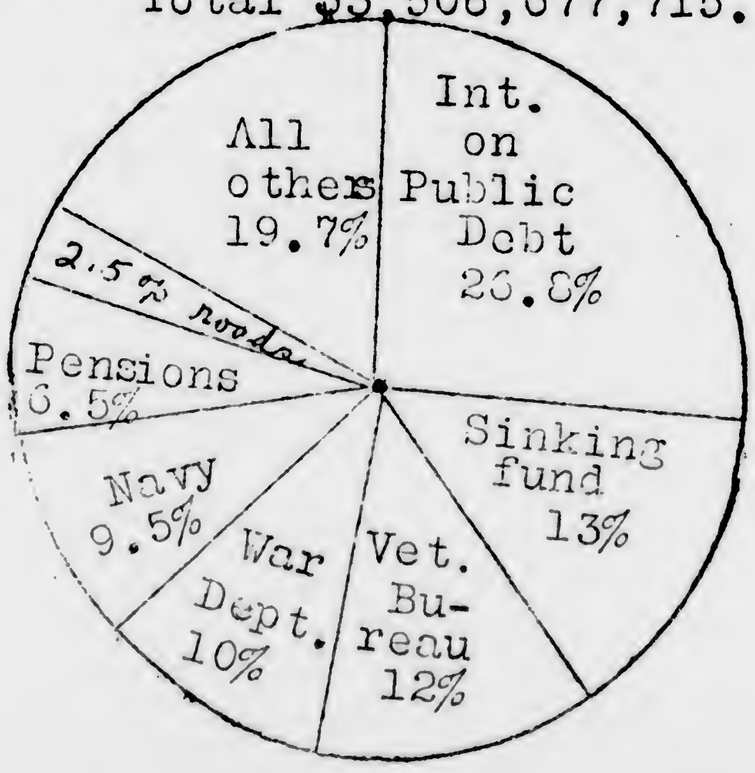
These are used to emphasize the geographical distribution of population, farm products, industries, etc. Three types are common, the colored map, the dot map, and the crosshatch map. This kind of graph is used extensively in the atlas and in the census reports.

II. The Circle Graph.

The circle is a favorite means of comparing statistics. A circle graph shows the relation of each part to all the other parts and also to the whole. It is popular in representing the distribution of a whole such as a dollar of income, taxes, budget, etc. The circle represents the whole of any thing. Radii are drawn to divide the circle into areas representing the various parts.

Circle graphs depend on the fact that the sum of all the angles about the center is 360°. This is divided by means of a protractor into parts so as to express the numerical relations. Thus $\frac{1}{10}$ of the total amount would be represented by an angle of 36°; $\frac{1}{4}$ of the whole would be represented by 90°, etc.

U. S. Government Expenditure (1924)
Total $3,506,677,715.



Exercise.

1. A man's salary is $2000 per year. He spends $480 for rent, $400 for food, $200 for clothing, $150 for light and heat, $100 for benevolence, $300 for all other expenses. The remainder he puts in the bank. Construct a circle graph of this problem.

2. Each day a freshman studies and recites 9 hours, sleeps 7 hours, spends 2 hours at his meals, uses 3 hours to earn money and 3 hours for recreation. Represent the day's activities by a circle graph.

3. Of every dollar which a certain town spends 22 cents goes to the schools, 15 cents to the streets. 10 cents to the police, 12 cents to the health department, and the remainder to overhead and upkeep. Make a circle graph showing the tax payer how his dollar is used.

III. The Bar Graph.

One of the simplest and most useful of graphs is the bar graph. This consists of straight, heavy lines or bars, either horizontal or vertical, whose lengths represent the relative magnitude of the quantities compared. It is suitable to represent numbers of the same kind which have no dependence on each other in order to show their relative magnitudes. In constructing a bar diagram the following directions should be observed:

1. Each graph should have an appropriate title neatly written where it is easily seen, preferably at the top.

2. All letters or figures used should be so placed that they can be easily read from the lower side or the right hand side of the sheet.

3. The bars should be constructed to a scale so chosen that bars of convenient length could be drawn to represent the largest and smallest numbers. Squared paper can be used to much advantage.

4. Bars should start from the same line. The zero of the scale should appear on the graph.

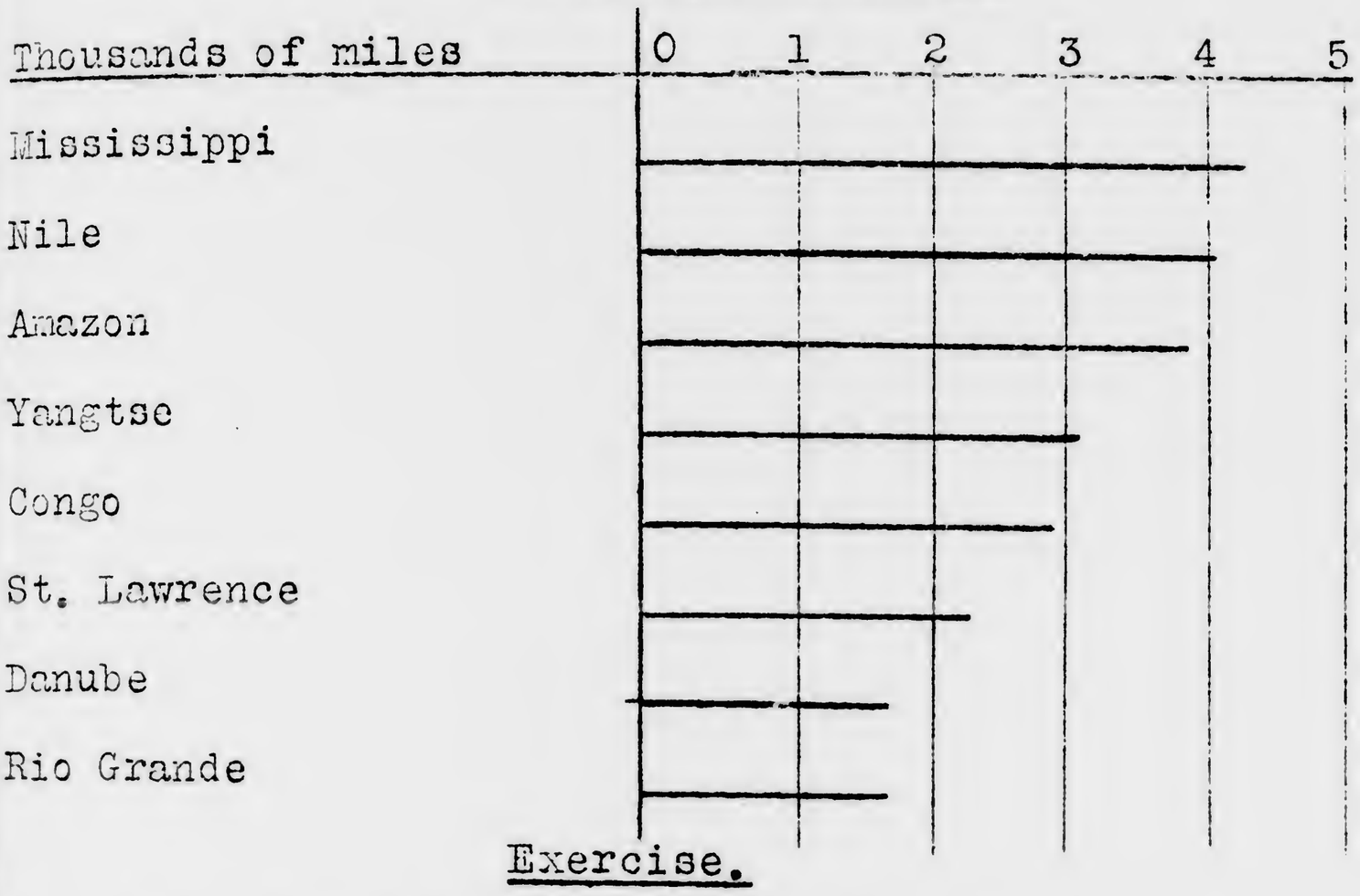
5. The bars should be of uniform width. The spaces between the bars are usually of the same width as the bars.

6. Each bar should be designated.

7. Accuracy is the most important consideration.

Show that the following horizontal bargraph reveals more clearly the relative lengths of rivers than the table given under the tabular method at the beginning of this chapter:

The Longest Rivers.



Exercise.

1. Construct a horizontal bar diagram showing the total value of the principal farm crops in the United States in 1923(given in millions of dollars)

Corn	$2,222	Potatoes	$229
Cotton	1,563	Sugar	2,086
Hay	1,391	Tobacco	299
Oats	539	Wheat	725

2. Construct a bar graph to show the exports of the United States to principal countries (1919) given in thousands.

Argentina	$224,718	Britain	$2,746,908
Canada	801,633	Italy	1,428,468
China	146,897	Japan	381,800
France	1,569,723		

3. Construct a bar graph comparing the ten largest universities in the United States (see World Almanac)

Bar diagrams may be used to show <u>several features</u> at once or to compare component parts.

4. Make a double horizontal bar chart (one bar black and the adjoining bar shaded) showing the value of products in the leading mining industries in the United States in 1909 and 1919 (given in millions of dollars).

Industry	1909	1919
Anthracite coal	149	364
Bituminous coal	428	1146
Copper	135	181
Iron ore	107	218
Lead and zinc	31	76
Petroleum	185	932

5. Make a subdivided bar chart to set forth the wages in manufacturing industries of the United States. Divide each bar to represent the proportion of men, women, and children.

Weekly wages	Men 16 and over	Women 16 and over	Children	Total
Under $6	516,185	526,206	151,834	1,194,225
$6 - $8	600,014	300,285	7,189	907,488
8 - 10	893,715	148,660	779	1,043,154
10 - 12	654,435	54,340	83	708,858
12 - 15	714,816	26,207	13	741,036
15 - 20	609,797	8,516	1	618,314
20 - 25	170,571	1,273	.....	171,844
25 - over	85,005	397	.....	85,402

6. Make a horizontal bar chart of the number of acres of land of United States devoted to crops of various kinds in each of the two years as represented in the following table:

Crops	First Year	Second Year
Cereals, other seeds	189,075,000	204,560,000
Flowers, small fruits	378,000	370,000
Sugar crops	790,000	1,285,000
Tobacco, cotton	87,067,000	106,000,000
Vegetables	5,630,000	7,073,000

7. Make a horizontal bar diagram showing the relation between the native born and foreign born population of 1890, 1900, 1910, and 1920.

V. The Vertical Bar.

The vertical bar diagram is particularly suitable for showing time variation and frequency distributions.

8. Draw a vertical bar diagram showing the population of the United States.

Year	Persons	Year	Persons
1820	9,638,000	1880	50,156,000
1830	12,860,000	1890	62,948,000
1840	17,069,000	1900	75,955,000
1850	23,192,000	1910	91,972,000
1860	31,443,000	1920	105,711,000
1870	38,558,000		

TIGHT BINDING

9. Make a vertical bar chart showing the per cent of the total population of the United States living in rural communities and in urban communities.

Year	Rural population	Urban population
1880	71.4	28.6
1890	64.6	35.4
1900	50.0	40.0
1910	54.2	45.8
1920	48.6	51.4

10. The grades in a class of fifty students are as follows: 5 students received E; 8, D; 20, C; 12,B; 5, A. Represent the grades by a vertical bar graph.

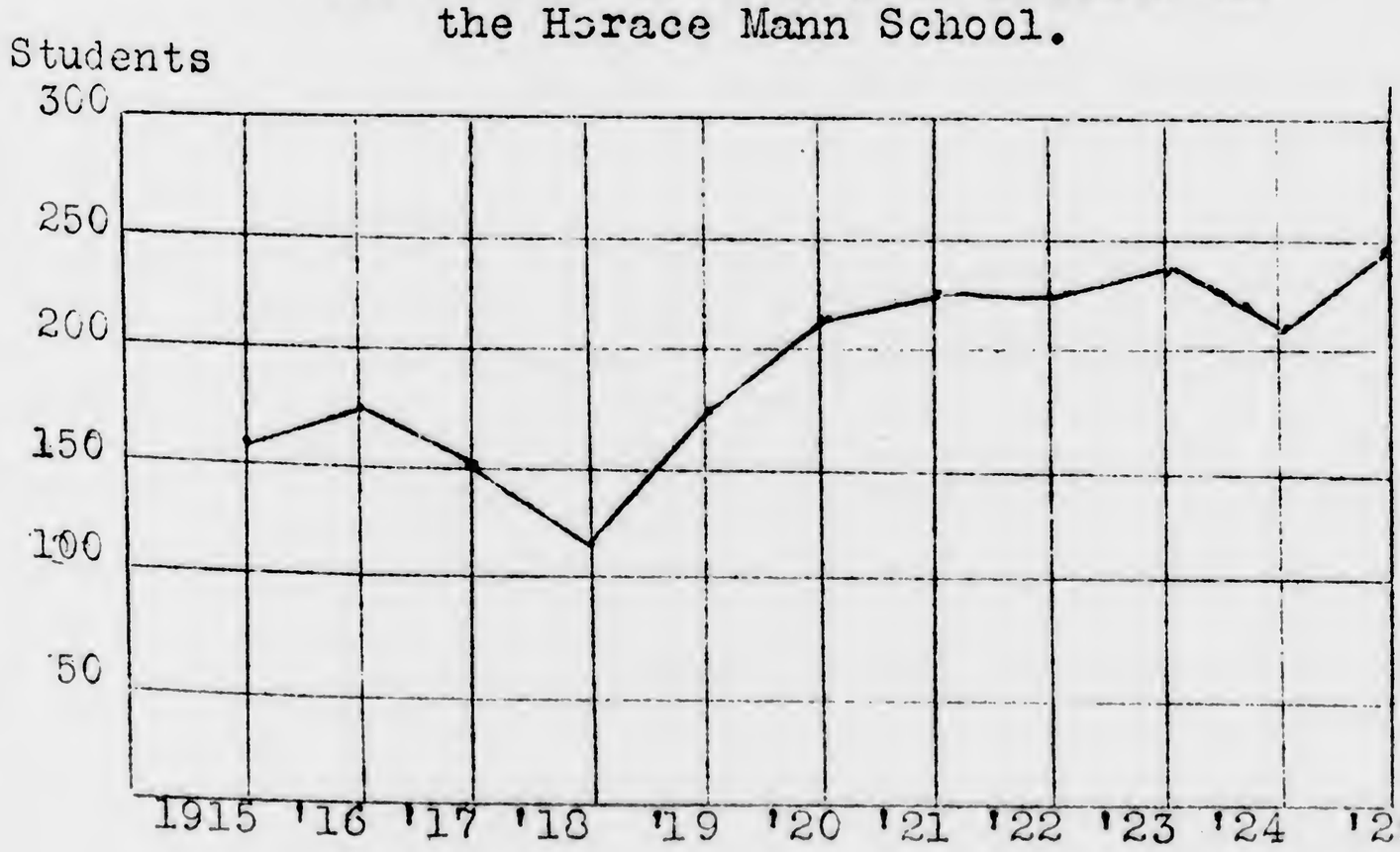
V. <u>The Broken Line Graph.</u>

When we are especially interested in the way quantities change from time to time, it is the ends of the vertical bars that hold our attention. These changes are more readily seen if the upper end points of the bars are joined by a series of straight lines called a <u>broken line</u>. After a little practice only the end points need to be indicated.

The broken bar graph is used in comparing a set of values which extend through a given time or to represent a set of values partially dependent upon another.

Example. Interpret the following graph :

The Number of Students Enrolled in the Horace Mann School.



Exercise.

1. Represent graphically the grades made by a student in mathematics in eight successive weekly tests. 70, 65, 75, 80, 75, 80, 85, 85.

2. Represent by a broken line graph the average farm price per bushel of corn on December 1, given in cents.

Year	Cents	Year	Cents	Year	Cents
1913	69	1917	128	1921	42
1914	64	1918	136	1922	66
1915	58	1919	134	1923	73
1916	89	1920	67		

3. Draw the graph of the following table showing the average weight of boys compared with their height:

Height in Inches	Weight in Pounds	Height in Inches	Weight in Pounds	Height in Inches	Weight in Pounds
48	53.5	53	69.0	58	87.5
49	55.5	54	72.5	59	91.5
50	59.5	55	77.5	61	95.0
51	63.0	56	79.5	61	99.5
52	66.0	57	83.5	62	105

Comparison of Graphs.

Sometimes it is interesting to compare two graphs. This is done by drawing them on the same figure using different colors or different kinds of lines.

4. Two flocks of hens 25 each made average daily egg records one year as follows:

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept	Oct.	Nov.	Dec.
Leghorns	8	13	20	23	22	18	17	15	10	5	2	5
Plymouth Rocks	7	10	18	22	23	20	16	13	8	3	2	4

Make a graph of this egg laying contest using a black line for leghorns and a dotted line for plymouth rocks.

5. A certain salesman made an average number of calls per day and took an average number of orders as follows:

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Calls	11	12	14	15	16	14	15	13	15	16	17	14
Orders	9	9	11	10	12	10	11	9	10	10	8	9

Construct a graph to show the salesman's activity and efficiency.

Leveling.

Leveling is the art of determining and representing the changes in elevation of the surface of the ground. The instruments used are the level, leveling rods, and tape. The principle is to find the elevation of each point above some base or datum. A graph of the elevations is then made to show the profile, or vertical section of the line of levels. Since the vertical distances are small compared with the horizontal distances, the paper is ruled with small divisions between the vertical lines and large divisions between the hor izontal lines.

6. A farmer runs a level line for a ditch. The following level readings in inches were taken at stations 5o feet apart.

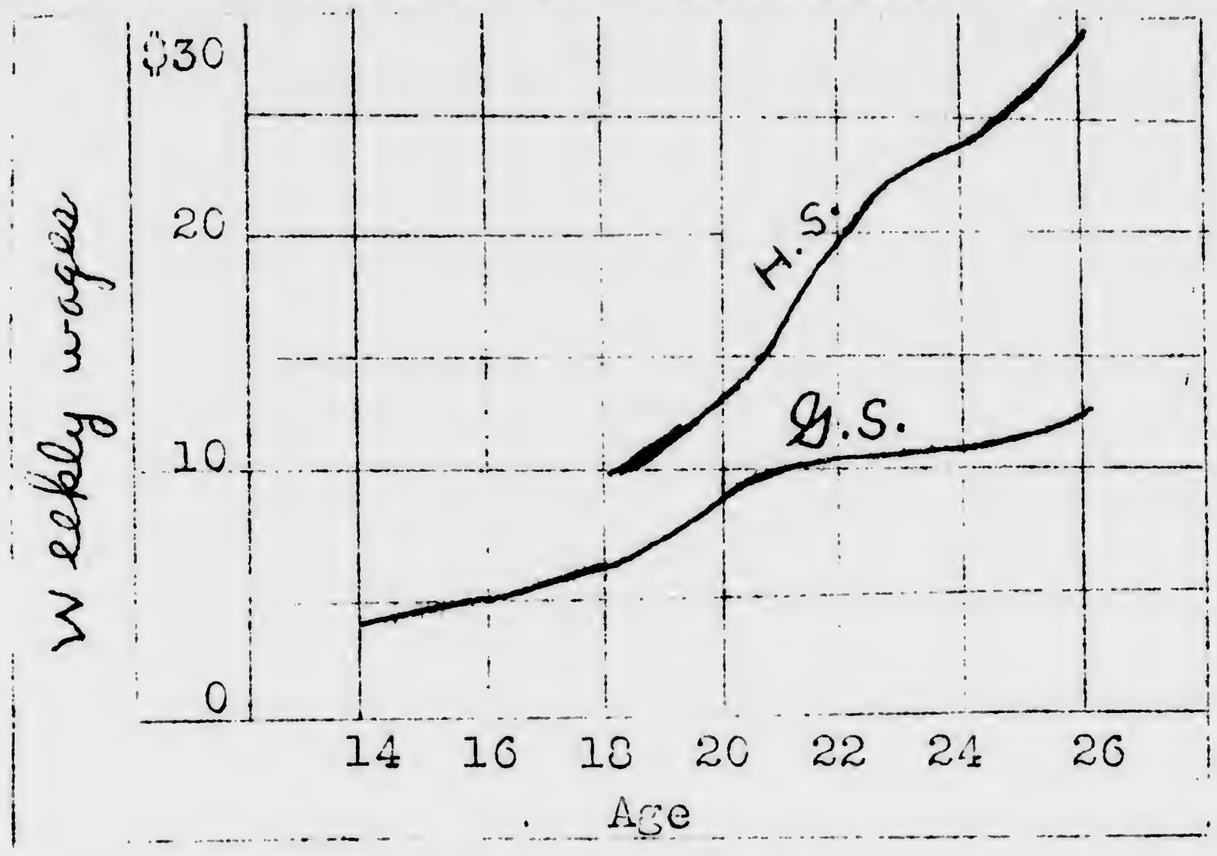
Station number	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Elevation	30	33	37	40	45	41	39	38	36	34	39	42	35	30	25

Make a profile of this level. Establish the grade line by finding the depth at each stake so that the ditch will be one foot deep at station O and slopes gradually to a depth of two feet at the last station.

VI. The Curved Line Graph.

This graph is constructed like the broken line except that after the points have been located they are joined by a smooth curve instead of straight line segments. The smooth curve emphasizes a continuous change from one point to another. It is used when the changes are gradual and continuous, such as temperatures, cost of living, population. Since the curved line graph is continuous we can make interpolations.

The following graph shows a contrast in the average weekly earnings for boys who leave school at the age of 14 with weekly earnings of boys who leave school at the age of 18.



Exercise.

1. Draw a curved line graph of the population of United States as given in exercise 8 under the bar graph.

2. Draw a graph of the following table which gives the standard pressure in inches on the mercury barometer for various heights above sea level:

Height in thousands of ft.	0	6	12	18	24	30	36
Pressure in inches	30.0	23.8	19.0	15.0	11.8	9.5	7.5

3. Draw a graph displaying the data in the following table which shows how many out of an average group of 1000 persons will be living at various ages:

Age	Number	Age	Number	Age	Number
0	1000	40	700	75	246
10	824	50	625	80	137
20	801	60	511	85	60
30	758	70	328	90	19

4. A certain automobile decreases in value each year by 20 % of its value at the beginning of the same year. Represent graphically the decreasing values in 6 successive years of a car costing $1200 when new.

5. Draw the temperature graph for a certain summer day taken hourly from 6 A. M. to 6 P. M.

6 A. M.	74°	10:00	80°	2:00	93°
7:00	75°	11:00	87°	3:00	84°
8:00	80°	12:00	91°	4:00	67°
9:00	83°	1 P.M.	93°	5:00	70°

6. Draw the temperature graph for a cold winter day.

6 A.M.	-7°	12:00	6°	6:00	18°
8:00	-4°	2 P.M.	15°		
10:00	0°	4:00	12°		

7. Does a graph ever show facts that are not shown by a table of values? Illustrate your answer.

8. The temperature of water in a pot at various times (T minutes) after being put on the stove is given below:

T	0	1	2	3	4	5	6	7	8
C	22°	36	50	64	78	92	100	100	100

Plot the graph. Should the graph be a broken line or a smooth curve? When did the water begin to boil?

9. The following gives the values of A and sine A:

A =	0°	30°	60°	90°	135°	150°	180°	210°	270°	330°	360°
sinA =	0	$\frac{1}{2}$	.86	1	.7	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0

Plot the graph. Find the values of sin 45°, sin 120°, sin 240°.

The Mathematical Graph.

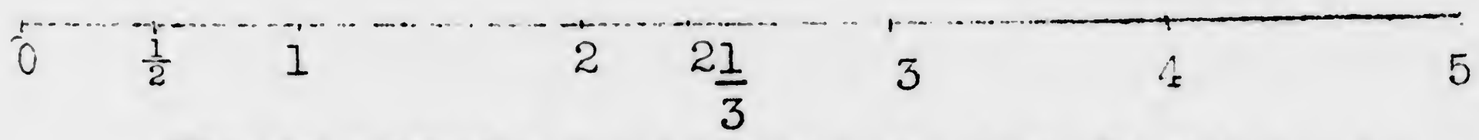
The graphs so far discussed represent independent values of the same kind or sets of values partially dependent on other sets. For this kind of relation no equation or formula could be made. The graphs of such relations are called statistical graphs.

There is another kind of problems in which the quantities are related in some definite way. When one set of values is entirely dependent upon another set so that the relation can be expressed by an equation or a formula, the mathematical graph, or functional graph, is then used.

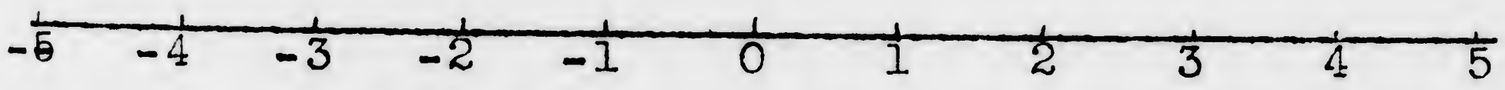
The process of constructing mathematical graphs does not differ very much from the method of constructing statistical graphs. In order to make it clear we shall first explain the mathematical scales and the lines of reference.

Scales.

In mathematics we frequently speak of the arithmetical scale and the algebraic scale. The arithmetical scale begins at zero and extends indefinitely in one direction including all the positive real numbers. Thus



The algebraic scale extends indefinitely in opposite directions from zero, with the positive numbers to the right and the negative numbers to the left.

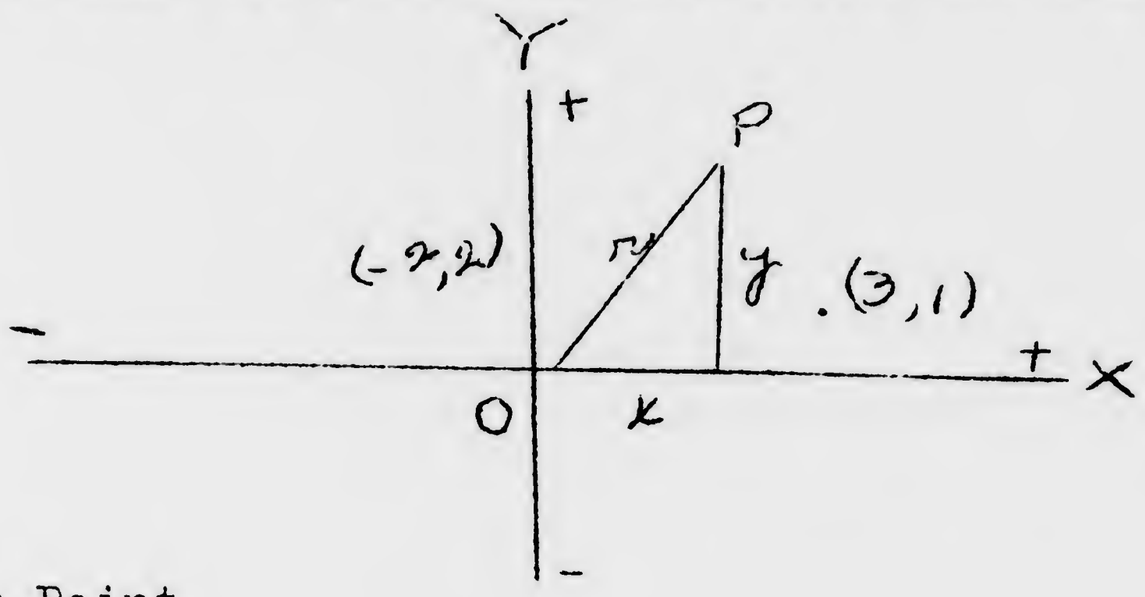


Rectangular Coordinates.

Two algebraic scales intersecting at the zero points at right angles to each other may be taken as lines of reference or axes. We shall usually think of one of them as horizontal and call it the x-axis, and call the other the y-axis. The point of intersection O is called the origin.

To locate a point in the plane it is merely necessary to give its distance to the right or left of the y-axis, and its distance above or below the x-axis. The distance of any point from the y-axis is called the abscissa of the point or its x. Its distance from the x-axis is called the ordinate of the point or its y. The abscissa and ordinate of a point together are called its coordinates.

Abscissas of all points to the right of the y-axis are positive, and of those to the left are negative. Ordinates above the x-axis are positive, and below the x-axis are negative.



Plotting a Point.

A point is designated by writing its abscissa and ordinate in a parenthesis with the abscissa first and the ordinate second. Thus (3,1) means the point whose abscissa is 3 and whose ordinate is 1. In the same way (-2,2) means the point whose abscissa is -2, and whose ordinate is 2, and so on. To plot a point is to locate it when its x-distance and y-distance are known.

Functions.

A Constant is a quantity which has a fixed value throughout a single discussion.

A Variable is a quantity that may take on different values in a discussion.

Function. When two variables are so related that one of them depends upon the other for its value, the first is called a function of the latter. For example, the circumference of a circle is a function of the radius. This may be expressed by the formula $c = 2\pi r$. In this formula 2 and π are constants because they have fixed values. But c and r are variables because they can be made to take different values. Thus when $r = 1$, $c = 2\pi$; when $r = 3$, $c = 6\pi$, and so on.

Dependent and Independent Variables.

Since we think of r in the formula $c = 2\pi r$ as taking on various values without regard to c, we speak of r as the independent variable. On the other hand, since the value of c is automatically fixed when some particular value is assigned to r, we speak of c as the dependent variable. It is customary in plotting to represent the independent variable on the x axis, and the dependent variable, or function, on the y axis whenever this distinction can be made.

Notation of Function.

The value of each of the expressions $4x - 2$, x^2, $4x^2 - 5x + 1$, etc., is known as soon as the value of x is known. Since their values depend upon the value of x, they are functions of x.

When we wish to refer to a given function of x without writing it all out, it is customary to use the symbol f(x), which is read either "function of x" or merely "f of x".

This notation is useful when we wish to evaluate an expression. It is understood that f(a) means the value of f(x) when a is substituted for x. That is, if

$$f(x) = 4x^2 - 5x + 1$$

$$f(a) = 4a^2 - 5a + 1$$

$$f(2) = 4(2)^2 - 5(2) + 1 = 7$$

$$f(-1) = 4(-1)^2 - 5(-1) + 1 = 10$$

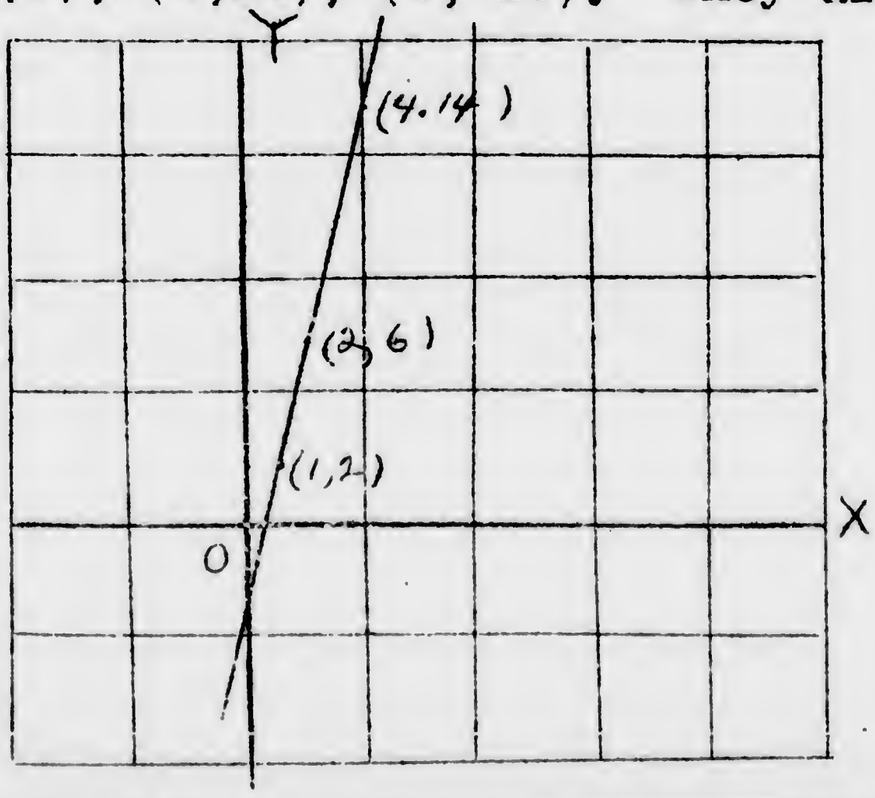
$$f(0) = 4(0) - 5(0) + 1 = 1.$$

Graph of a Function.

Let us graph the function $4x - 2$, which represents the perimeter of every rectangle whose length exceeds its width by one unit. Here we have two variables, x and $4x - 2$. By substituting various values for x we ontain corresponding values for $4x - 2$, which is f(x). Thus

If x =	1	2	4	5	7	10
Then f(x) =	2	6	14	18		

These pairs of values may be plotted, each pair giving one point on the graph. Use the horizontal scale for the x values and the vertical scale for the values of f(x). Locate the points (1,2), (2,6), (4,14), (5, 18). They all seem to lie on a straight line.



Graphs of Equations.

A. Equations of the First Degree.

We can put the example just given in a slightly different form. If we denote the variable 4x - 2 by a single letter y, we have

$$y = 4x - 2.$$

Then y stands for the function 4x - 2. The graph, of course, will be exactly the same as before. The equation $y = 4x - 2$ may be put in the form $y - 4x = -2$ or $4x - y - 2 = 0$, but the graph will not be changed.

How to Draw the Graph of an Equation of the First Degree.

Draw the graph of $4y - 3x = 6$.

(1) Solve the equation for y, $y = \frac{3x + 6}{4}$

(2) Give values to x and compute corresponding values for y and build a table thus:

Let x =	0	2	6	9
Then y =	$1\frac{1}{2}$	3	66	

(3) Plot the points corresponding to the pairs of numbers of the table.

(4) Join these points by a line.

The graph of every equation of the first degree is a straight line. For this reason first degree equations are called Linear Equations.

Short Method of Drawing the Graph of a Linear Equation.

A straight line is determined or located by two points. Therefore, two points only are needed to draw the graph of a linear equation. Any two points far enough will do. The easiest points, however, are those where the line cuts the axes. The distances of these points from the origin are called the intercepts of the graph. If we let y = 0 in any equation, we find the x intercept. To find the y intercept we let x = 0 and take the resulting value of y. The line through the intercepts is the graph required. In the example $4y - 3x = 6$, if y is 0, x is -2; hence, the line cuts the x axis at the point (-2,0). If we let x = 0, then $y = \frac{3}{2}$; the line cuts the y axis at $(0, \frac{3}{2})$.

This shorter method requires only the following brief table:

x =	0	?
y =	?	0

Let us now represent the relation between the Fahrenheit and Centigrade thermometers.

(1) The verbal statement of this relation is as follows: "When water freezes, the Centigrade thermometer reads zero and the Fahrenheit reads 32°. When water boils the Centigrade reads 100° and the Fahrenheit reads 212°. There is a constant relation between a Centigrade degree and a Fahrenheit degree".

(2) This verbal statement may be expressed by the equation

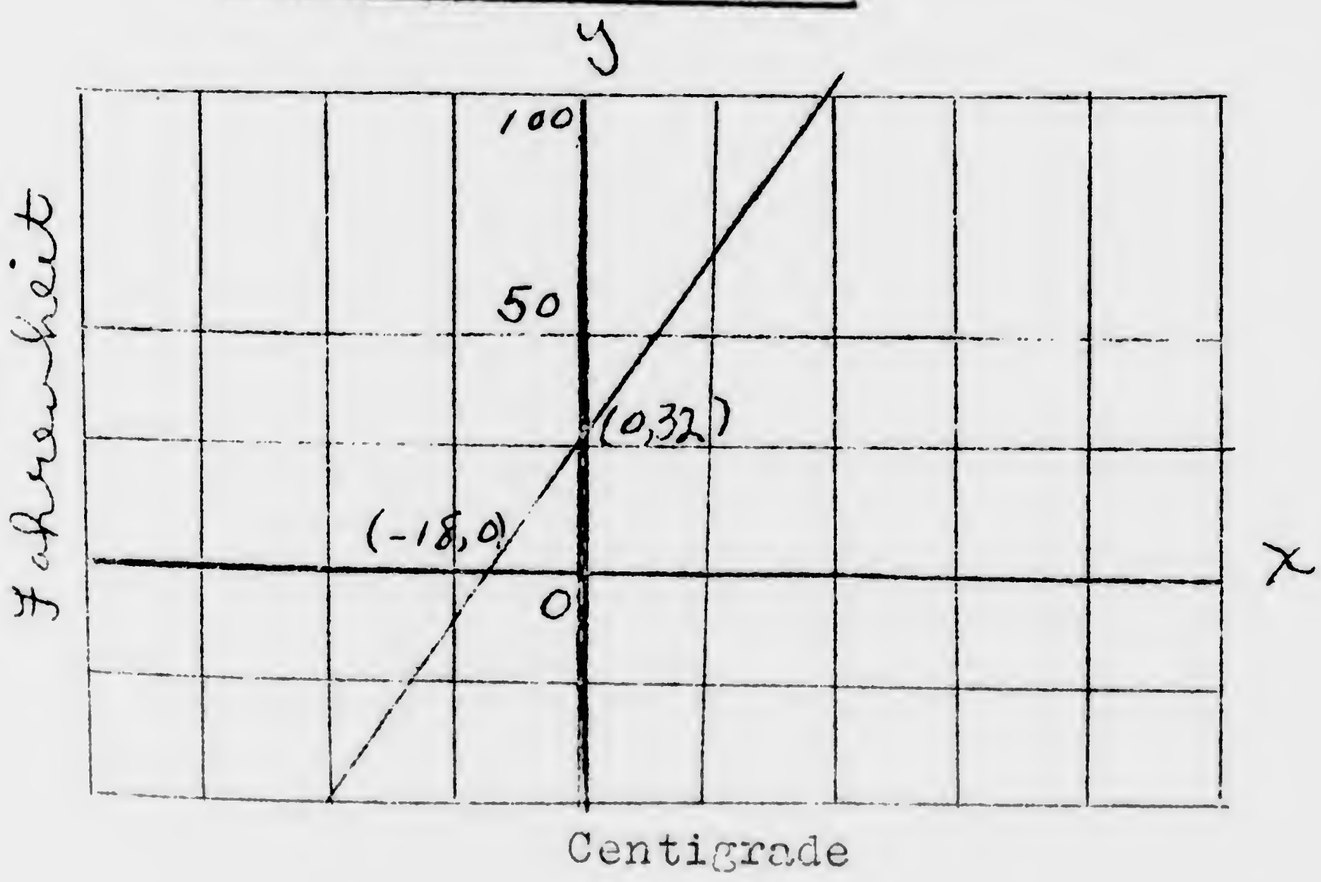
$$F = \frac{9}{5} C + 32.$$

By means of this formula we can translate readings from one scale to another.

(3) It is easier, however, to show the relation by graph. We first make a table of values or we find the intercepts as follows:

C =	0	-17.8
F =	32	0

Plot these intercepts and draw a straight line through them.



By using the graph we can translate one temperature to the other without any calculations.

The thermometer equation is of the form
$$y = mx + b$$

where $m = \frac{9}{5}$ and $b = 32$. Every equation of the first degree in two unknowns can be put in this form by solving it for y. The multiplying constant m is the slope of the line, for the greater the numerical value of m the steeper is the line. When m is positive the line slopes up to the right. When m is negative, the line slopes down to the right. If m is zero the line is horizontal.

The constant term b stands for the y intercept. If b is positive the line crosses the y axis above the origin; if b is negative the line crosses the y axis below the origin. If b = 0 the equation is of the form y = mx and its graph passes through the origin because when x = 0, y = 0. Can we plot equations of the form y = mx by the intercept method? Why?

Exercise.

1. Plot the following points:
   (2,-3), (-2,1), (-3,4), (0,-5), ($2\frac{1}{2}$, -6), (0,0).

2. The abscissa of a point is 4. Is the point definitely located? Why?

3. Where are all the points for which x = 3? x = -1? y = 5? x = 0? y = 0?

4. Compare the process of plotting points with the numbering of houses in a city; with locating a place on the map.

5. $f(x) = 3x^2 - 5x + 2$. Find f(0); f(2), $f(3\frac{1}{2})$; f(-1); f(-2).

6. Draw the graphs of the following equations:
   (a) $x + y = 3$ (b) $x - 2y = 6$ (c) $5x - 3y = 15$
   (d) $3y - 2x = 10$ (e) $y = 5x$ (f) $x + 3y = 0$

7. Nails sell at 7 cents a pound. Make a table of the prices of different weights and draw a sales graph. How many pounds will 25 cents buy?

8. A car is moving uniformly at the rate of 25 miles an hour. Draw a graph showing the distance traveled in any given time less than 5 hours.

9. Make a table showing the relation between inches and centimeters and draw the graph. One inch equals 2.54 centimeters. Find from the graph how many inches there are in 15 c. m., in 40.5 c. m.

10. Draw a graph of the simple interest on one dollar at 6% for a number of years. Draw also a graph for the amount of one dollar.

11. The cost of sending a package by parcel post into a certain zone is three cents plus two cents for every ounce. Write a parcel post formula and draw its graph.

12. The cost for setting up the type for printing a certain circular is $2.50. The price is 40 cents per hundred for copies. Write a formula for the cost of any number of cards and draw the graph.

13. If a point is on a line do the values of its x and y satisfy the equation of the line?

14. If a point is not on a line do the values of its x and y satisfy the equation of the line?

15. If the values of the x and y of a point satisfy the equation of a line, is the point located on the graph of the equation?

16. Is the point (1,3) on the line whose equation is $3x + y = 6$? Is (2,0)? Is(5,-9)? Is (4,-5)?

17. Can you plot an equation with three unknowns like $2x - y + 3z = 0$?

18. The relation between equations and lines(between algebra and geometry) was found by a Frenchman named Descartes. Write a brief sketch of his life.

B. Graphs of Non Linear Functions.

We shall now discuss the construction and interpretation of graphs of equations not of the first degree. These graphs are not straight lines but curved lines.

I. Graph of a Quadratic Function.

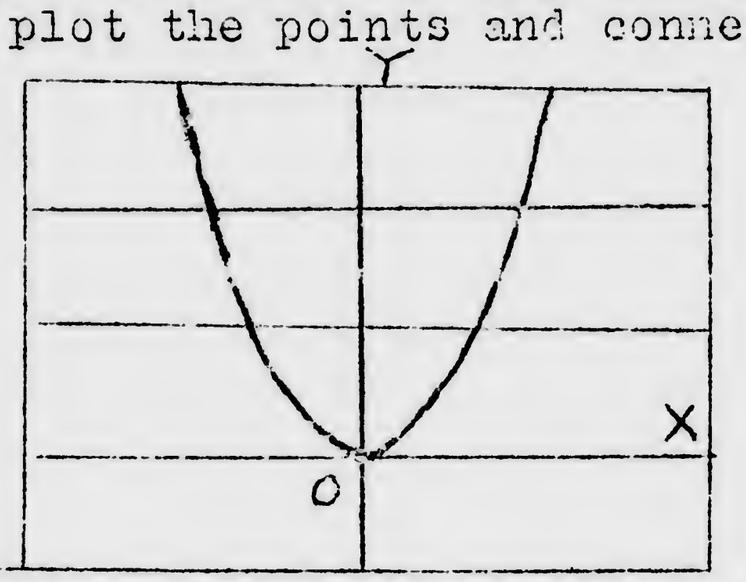
One of the most common of non-linear functions is the function of the second degree. The general type of this function is

$Ax^2 + Bx + C$. In this type form, B or C or both B and C

may be zero, but A cannot. That is, $5x^2 + 2x$, $4x^2 - 1$, and $3x^2$ are quadratic functions of x.

The simplest case is $f(x) = x^2$ which may be written $y = x^2$. To plot this equation we make a table of values by giving x negative and positive values and finding the corresponding values of y. Then we plot the points and connect them by a smooth curve.

X =	-2	-1	0	1	2	3
Y =	4	1	0	1	4	9



The graph is a parabola. It shows the relation between a number and the square of that number. How does this graph help us to determine the square root of a number?

In a similar manner we may graph any quadratic function of the type $Ax^2 + Bx + C$ such as $x^2 - 2x - 6$, $3x^2 + 4x + 1$, $x^2 - 4x$. The graph of a quadratic function of x is always a parabola. The parabola may be in different positions with respect to the axes.

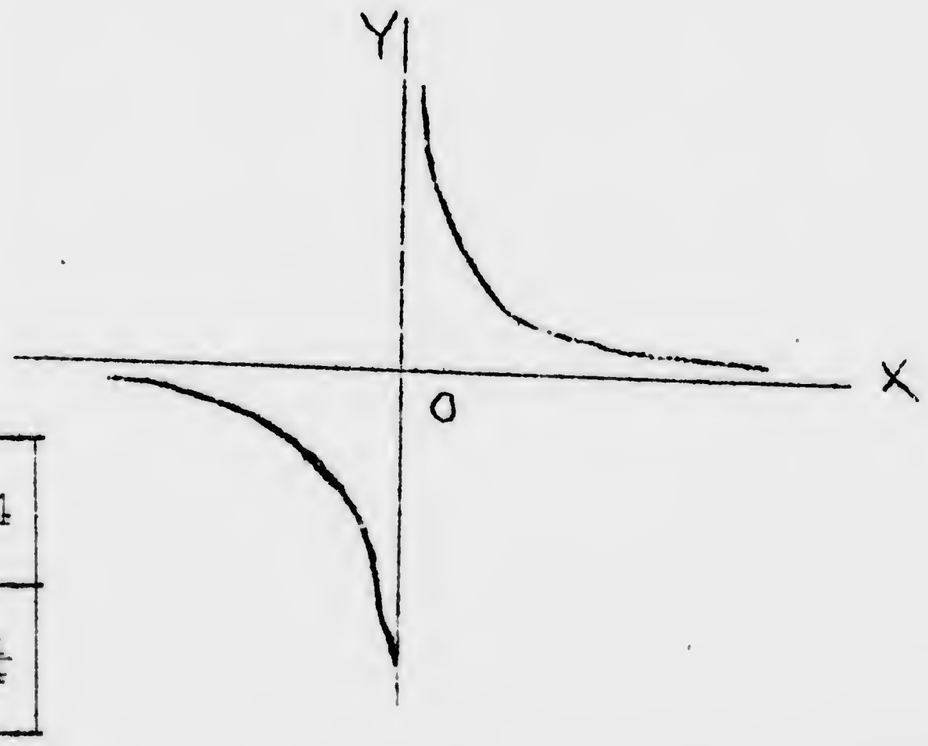
The parabola is a common curve. The path of a projectile is a parabola. The arch of a bridge and the crown of a pavement are often made parabolic. Reflectors of searchlights and telescopes are parabolic. The graphs of many scientific laws are parabolas.

II. The Equilateral Hyperbola.

Another common and useful curve is the equilateral hyperbola. Its equation is $xy = K$, where K is any constant not zero. The graphs of the equations $xy = 1$, $xy = 5$, $xy = -10$, are equilateral hyperbolas.

Draw the graph $xy = 1$, or $y = \frac{1}{x}$, or $y = x^{-1}$

x	$\frac{1}{4}$	$\frac{1}{2}$	11	2	4
y	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$



x	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4
y	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$

The equilateral hyperbola is often used in scientific work and in representing economic data. If two variables are so related that when one of them increases, the other decreases in the same ratio, they are said to vary inversely, or to be inversely proportional. (See chapter on Variation) The equilateral hyperbola is the graph of inverse variation. As x increases, y decreases. As the price of an article increases the demand decreases. The curve is used in educational experiments. For example, the longer the practice in drawing a certain figure, the shorter is the time per trial.

The Power Function.

It is very common for a quantity to vary as the power of another. The general equation is

$$y = a x^n.$$

When the exponent n is positive the curve is parabolic, when n is negative the curve is hyperbolic.

III. The Law of Growth.

Many quantities in nature grow at a rate directly proportional to the value of the growing quantity at any time. The number of bacteria in a culture increases every second at a constant rate proportional to the number of bacteria at the beginning of the second. The growth of timber in a large forest obeys the same law. The growth of bean plants within limited intervals follows the same law. This law is called the "law of organic growth". This is the way a sum of money would grow at compound interest if the interest is converted very often or continuously. For this reason the law is often called the "compound interest law".

When a quantity y grows at a constant percentage rate r (per hour, day, year, or other unit of time) its value after t units is

$$y = c e^{rt}$$

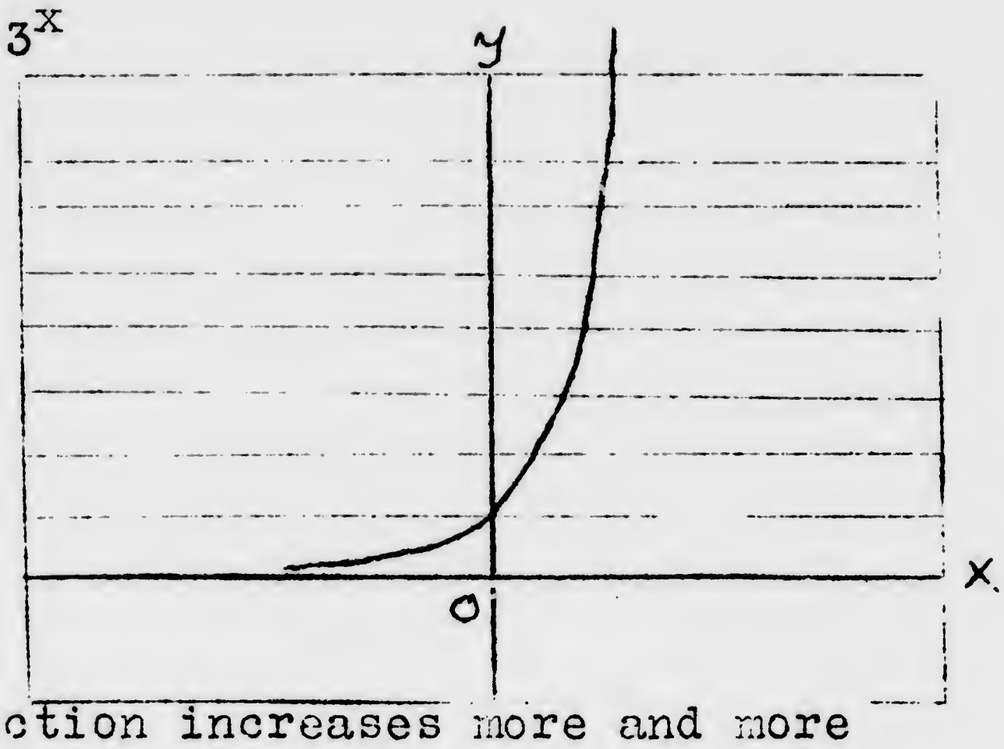
Where c and r are constants determined by the physical facts involved and e is a constant equal to 2.718.... This formula is a special form of the general exponential function

$$y = a b^x$$

It is called an exponential function because the exponent is a variable. The characteristic of the function, as previously noted, is that its rate of change at any point is always proportional to the ordinate or to the value of the function at that point.

Plot the equation $y = 3^x$

x	-2	-1	0	1	2	3	
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	



The graph shows that the function increases more and more rapidly.

When the quantity is decreasing instead of increasing, the proportionality constant r is negative, and the equation becomes

$$y = c\, e^{-rt}$$

As an example of a decreasing function, the decrease in the pressure of the air at the height h above sea level is proportional to h. Another example is the decomposition of radium, its rate of decrease being proportional to the quantity. Similarly in the case of a vibrating body, a noise dying down, and in certain electrical phenomena, the rate of decrease is proportional at any instant to the value of the function at the instant.

Exercise.

Plot the graphs of the following:

1. $y = 3x^2$.
2. $y = x^2 + 2$.
3. $y = 2 - x^2$.
4. $f(x) = x^2 - x - 2$.
5. $f(x) = x^2 - 2x + 1$.
6. $f(x) = x^2 + x + 1$.
7. $y = \frac{1}{2}x^2$.
8. $y = x^3$.
9. $y = \frac{1}{2}x^3 + 3$.
10. $xy = 4$.
11. $xy = -6$.
12. $y = 2^x$.
13. $y = e^x$.
14. $y = (1.04)^x$.
15. $y = 3^{-x}$.
16. The number of bacteria in a culture increases at the rate of 25% per hour. Find how many there were at anytime from an original 500. How many after four hours.

Note. Without the use of logarithms we can find the value of only very simple exponential functions.

C. Graphical Solution of Equations.

We have seen that the main purpose of constructing graphs is to represent relationships between related variables because graphs make the relationship more vivid and enable us to see some of the properties of the function that we might otherwise miss. We shall now learn how graphs help us to solve equations.

I. Graphic Solution of Equations in One Unknown.

Consider the quadratic equation in one unknown

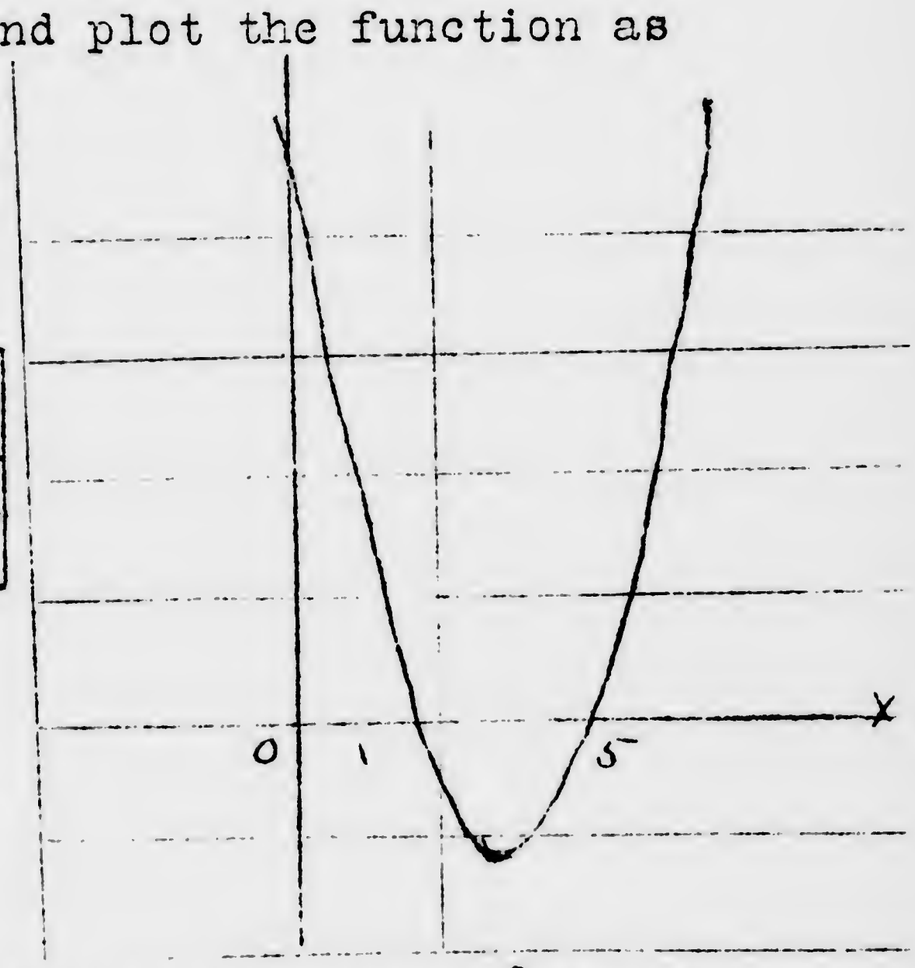
$$x^2 - 7x + 10 = 0$$

The expression $x^2 - 7x + 10$ may have any number of values depending upon the value of x. Therefore, it is a function of x; that is,

$$f(x) = x^2 - 7x + 10.$$

Make a table of values and plot the function as follows:

x	-1	0	1	2	3	$3\frac{1}{2}$	4	5	6
f(x)	18	10	4	0	-2	$-\frac{9}{4}$	-2	0	4



Does the graph show what value of x makes the function equal to 4? equal to -2? equal to 0?

The values of x which make the function $x^2 - 7x + 10$ equal to 0 are the solutions of the equation $x^2 - 7x + 10 = 0$. The values of x which make $f(x) = 0$ are the points at which the curve cuts the x axis. By the graph they are $x = 2$ and $x = 5$. Therefore, 2 and 5 are the roots of the equation.

If the graph is tangent to the x axis, what does it tell about the roots? If the graph does not cut the x axis, what does it tell about the roots?

In a similar way the graph of a cubic equation in one unknown (or of an equation in any degree) gives the roots of the equation where it cuts the x axis.

To solve an equation in one unknown graphically, follow these directions:

1. Make the right number zero, and draw the graph of the left number.

2. The points where the curve crosses the x axis give the values of x which make the function 0. These values of x are the roots of the equation.

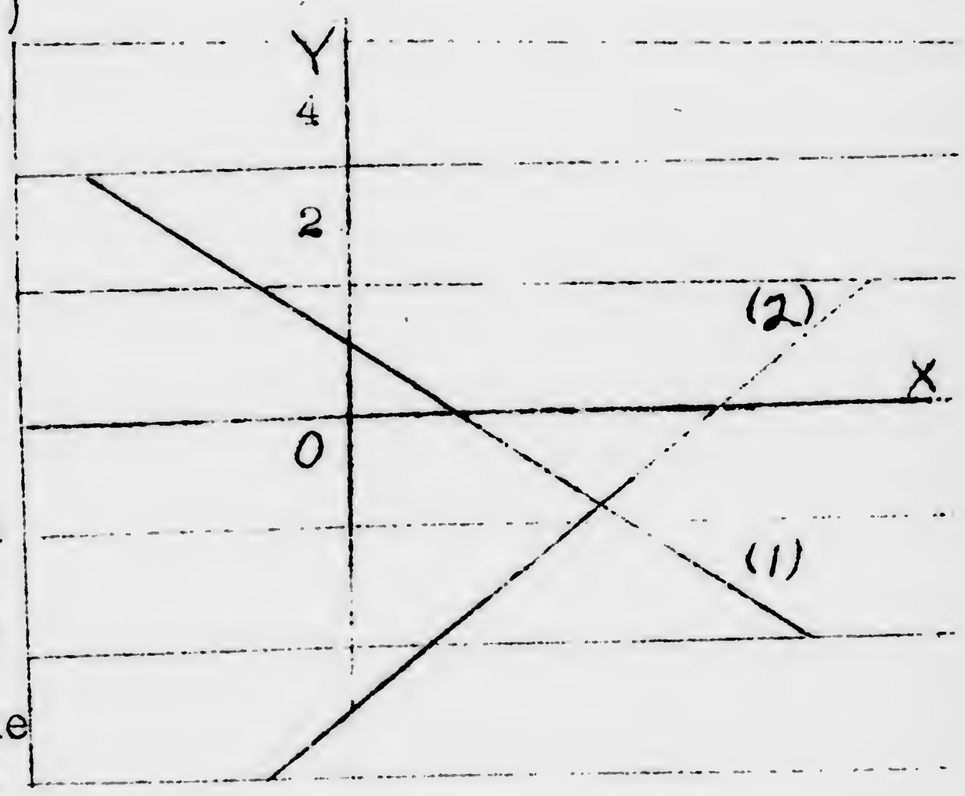
II. Graphic Solutions of Systems of Equations.

It should now be clear that:
(1) The equation of a graph is satisfied by the values of the x and y of any point on the graph.

(2) Any point whose x and y satisfy the equation is on the graph of the equation.

These two facts enable us to solve graphically a pair of equations which are satisfied at the same time (simultaneously) by the same pair of values.

For example, what pair of values will satisfy $2x + 3y = 4$ (1)
and satisfy also $3x - 4y = 23$ (2)



We construct the graphs of the two equations on the same axis. The coordinates of every point on line(1) are the values of x and y which satisfy the first equation. The coordinates of every point on line (2) are the values of x and y that satisfy the second equation. Since the point of intersection is on both lines, its coordinates satisfy both equations. The graph shows that the two lines intersect at the point (5,-2). Therefore, the solution of the system is $x = 5$, $y = -2$.

In the same way systems of quadratic equations may be solved graphically by the following

Rule.

(1) Draw the graphs of both equations on the same axes.
(2) The coordinates of each point of intersection are a solution of the system.
(3) If the graphs do not intersect, the roots are imaginary.

Exercise.

Solve the following equations graphically and check:

1. $x^2 - x - 12 = 0$
2. $x^2 - 5x - 6 = 0$
3. $2x^2 + 5x - 7 = 0$
4. $x^2 - 4x + 4 = 0$
5. $x^2 - 4x + 6 = 0$
6. $4x^2 + 4x + 1 = 0$
7. $3x^2 + 2x - 3 = 0$
8. $x^2 - 3x - 10 = 0$
9. $x^2 + 3x + 9 = 0$
10. $x^3 - 4x = 0$
11. $x^3 - 4x^2 + 3x = 0$
12. $x^3 - 5x + 3 = 0$

Solve the following systems of equations graphically and check:

13. $3x - y = 3$
    $5x + 2y = 16$
14. $6x + 8y = 26$
    $5x - 3y = 70$
15. $5x - 3y = 0$
    $15x + 12y = 75$
16. $2y - 7x = 38$
    $3y - 4x = 31$
17. $3r + 4s = 3$
    $12r - 5s = 3$
18. $x - 2y = z$
    $3y = 2x$
19. $x^2 + y^2 = 25$
    $x + 2y = 5$
20. $x^2 = 4y$
    $x + 2y = 4$
21. $xy = 6$
    $x + 2y = 8$
22. $y = x^2 - 5x + 4$
    $x - y = 4$
23. $y = x^2 - 5x + 4$
    $x - y = 5$
24. $y = x^2 - 5x + 4$
    $x - y = 6$
25. $y^2 - x^2 = 16$
    $5x + 3y = 0$
26. $x^2 = y + 4$
    $y^2 = -5x + 4$
27. $x^2 - 9 = 0$
    $x^2 + y^2 = 16$

Chapter XII

S T A T I S T I C S

In recent years educators, social scientists, and students of business and industry have been employing in a greater measure than ever before a method of study based upon the observation and analysis of facts. This accounts for the steadily increasing interest in the methods of statistics which facilitate the condensation, analysis, and interpretation of large masses of facts. It is, therefore, desirable that an educated man should have such an understanding of the elementary principles of statistics as will help him to appreciate scientific investigations and conclusions.

A. The Frequency Table.

In dealing with long tables of figures it is difficult to grasp their significance unless they are organized and condensed in a systematic way. It is, therefore, necessary to group the facts into classes of convenient size. The distribution of the facts over the several classes is called frequency distribution. A table showing the frequency distribution, that is, designating the classes and giving the frequencies of each, is called a Frequency Table.

For the purpose of illustration let us make a frequency table of the grades of 100 students in freshman mathematics given below:

75	86	66	86	50	78	66	79	68	60
80	83	87	79	80	77	81	92	57	52
58	82	73	95	66	60	84	80	79	63
80	88	58	84	96	87	72	65	79	80
86	68	76	41	80	40	63	90	83	94
76	66	74	76	68	82	59	75	35	34
65	63	85	87	79	77	76	74	76	78
75	60	96	74	73	87	52	98	88	64
76	69	60	74	72	76	57	65	67	58
72	80	72	56	73	82	78	45	75	56

Frequency Table of 100 Grades.

Score	Tabulation	Frequency
30-39	11	2
40-49	111	3
50-59	𝍸 𝍸 1	11
60-69	𝍸 𝍸 𝍸 𝍸	20
70-79	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 11	32
80-89	𝍸 𝍸 𝍸 𝍸 𝍸	25
90-99	𝍸 11	7
Total number of students =		100

The advantages gained in making a frequency table are obvious. It makes it easier for us to interpret the facts and to answer many questions concerning them.

The Steps in Making a Frequency Table.

(1) Determine the Range of the table by finding the difference between the highest value and the lowest value among the items. In this example the highest value is 98, and the lowest is 34. The range is 64. (In some cases, it is well to arrange the items according to rank. This may easily be done by writing each item on a slip of paper and arranging the slips in order).

(2) Determine the number of equal parts into which the range shall be divided for convenience in tabulation. These equal parts are called Class Intervals. In our example there are 7 class intervals of 10 units each. The size of the class interval and the number of intervals depend upon the size and nature of the table. In all grouping we assume that all the measures in a class interval are grouped at the mid-point of the interval and are equal to it.

(3) The Class Limits are the upper and lower extremes of a given class. Set the limits at such points as will lead to the greatest ease and accuracy of tabulation and computation.

(4) Arrange a sheet with three headings, score, tabulation, frequency.

(5) Read off the grades in the raw table and for each one record a mark as shown in the figure.

(6) Write the sum of the marks in each row in the frequency column. The sum of the frequencies should equal the number of items in the table.

Exercise.

1. Tabulate the grades of Table 1 using class intervals of 5 units.

2. Tabulate the same grades using class intervals of 15 units; of 20 units.

3. The following table gives the heights of 120 freshmen:

71.6	71.4	69.3	66.9	67.5	69.8	68.4	71.0	68.2	64.6
70.4	69.5	71.5	66.5	67.8	71.4	69.8	71.2	69.5	65.2
69.5	68.2	72.3	68.1	66.2	72.5	72.0	68.3	71.2	66.1
69.4	68.5	67.2	72.9	66.3	70.1	70.9	68.7	69.7	70.4
66.9	67.3	68.9	64.4	66.4	70.4	73.2	68.6	69.6	73.7
68.7	65.7	68.6	64.9	65.3	70.9	64.2	68.9	71.1	70.6
64.8	62	68.4	65.5	65.8	63.3	64.6	62.4	69.0	69.7
70.2	63.5	67.2	71.8	63.8	67.8	66.9	65.5	68.1	67.5
68.6	66.5	67.7	71.9	66.6	69.2	66.8	67.0	68.0	65.6
68.3	66.6	67.5	70.2	69.0	70.9	68.0	67.8	67.8	65.8
66.6	67.1	67.0	70.6	69.3	65.2	68.0	69.1	69.2	66.0
68.6	70.5	67.0	74.0	70.7	67.1	69.3	68.3	70.0	67.8

Is it easy to answer from this table such questions as these: (1) How many students were between 5 ft. 6 in., and 5 ft. 7 in.? (2) How many measured less than 5 ft. 4 in.? (3) How many were above 6 ft. tall? (4) What was the height of the tallest man? of the shortest man? (5) What per cent of the class measured between 6ft. 8 in. and 6 ft. 9 in.? (6) What is the most common height? and so on.

Prepare a frequency table using a class interval of one inch and then answer the above questions.

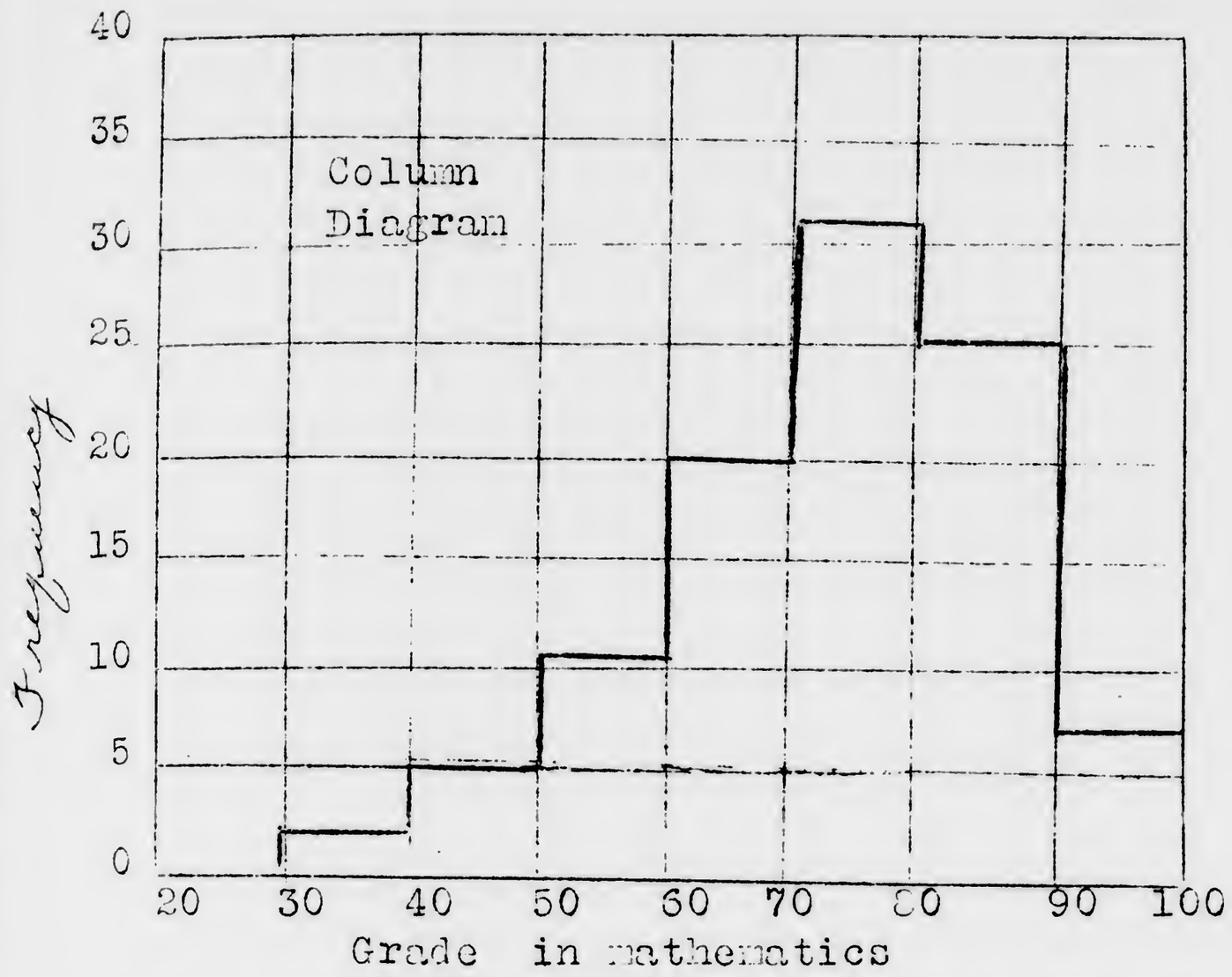
4. Prepare a frequency table using 2 inches as a class interval.

B. The Frequency Graph.

The interpretation of a frequency table is made easier by making a graph of the table. There are two kinds of graphs used to represent a frequency table, the column diagram and the frequency polygon.

(1) The Column Diagram (Histogram).

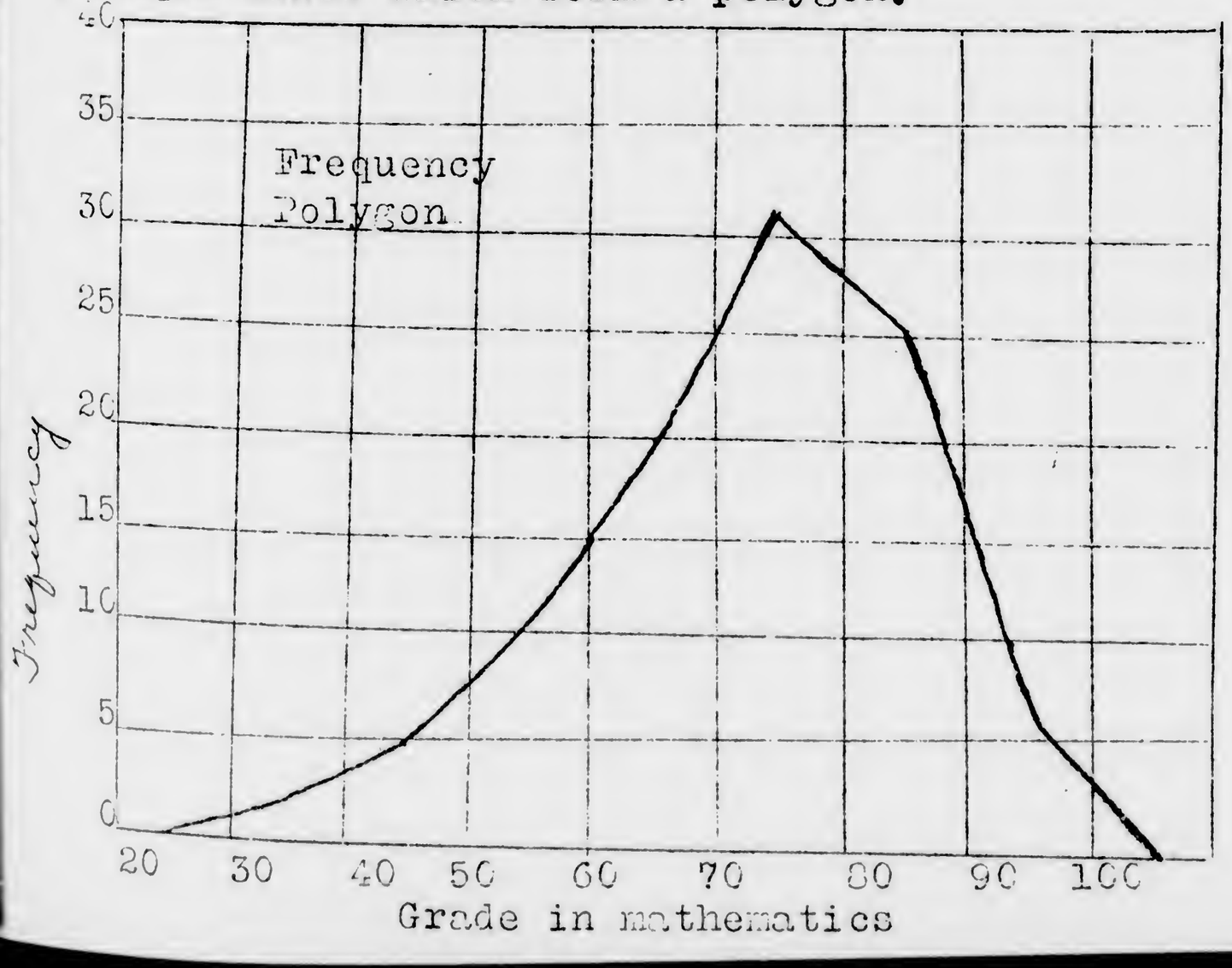
The Column Diagram is similar to the vertical bar graph. The class intervals are arranged on a horizontal scale, and the frequencies at a vertical scale to the left of the diagram. The base of each column is a whole class interval.



It is interesting to note that the area of a column diagram is proportional to the total number of individuals represented by the diagram.

(2) <u>The Frequency Polygon</u>.

In constructing a frequency polygon we assume that all the items in a class interval are at the midpoint of that interval and that they are equal to it in value. Then we plot the frequencies directly above the midpoints of the intervals. The points thus determined are connected by straight lines which form a polygon.



A frequency polygon can be constructed by joining with straight lines the midpoints of the tops of the column diagram.

Exercise.

1. Plot a column diagram for the student heights in exercise 3 under the frequency table when the interval is 1 inch, when the interval is 2 inches.

2. Plot a frequency polygon for the same.

C. The Normal Curve.

Normal Distribution.

If we measure the heights of a large number of persons, we shall find only a few very tall persons and a few very short persons. The majority of measures will be found concentrated near the middle.

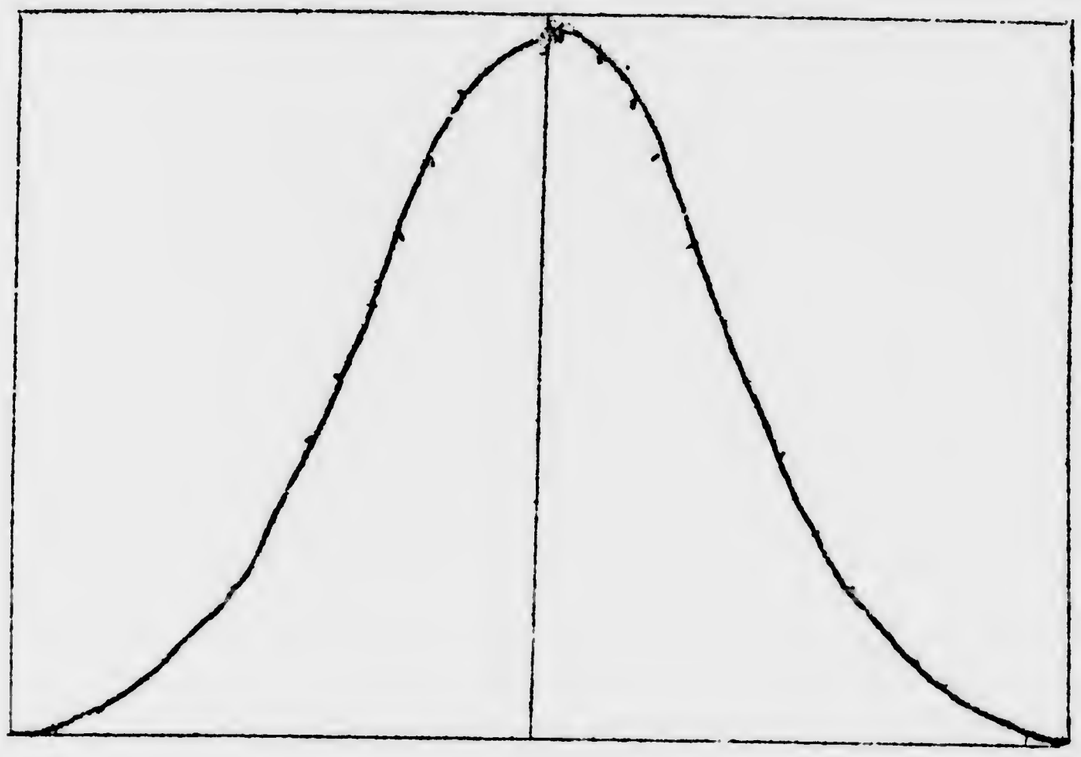
This is true of all human traits. Any physical measurement will show that the frequencies increase gradually to a maximum and then decrease in much the same way. If we measure ability to spell, to run, to jump, we shall find a few very good at it, a few very poor, with the majority in the middle.

The same is true in economic affairs. The weekly earnings of employees in a factory, the annual income of farmers, the price variation of an article, the number of students per teacher, all show the same tendency, although the distribution is not quite so regular.

The distribution of errors of observation follows the same plan. If fifty students were to measure carefully with a correct measure the length of a table, the chances would be that as many would give results too short as too long. Very small errors would be most common; the positive and negative errors would be equally frequent; and very large errors do not occur. Use is made of the compensation of errors in fine measurements, as in astronomy and surveying.

The same tendency is seen in nature. The length of leaves, the height of wheat plants, the length of ears of corn are grouped near a central measure and decrease gradually in both directions. This distribution comes closest to the ideal distribution of the mathematical law of chance and probability.

When the distribution is regular its graph approximates the ideal probability curve, called also the Normal Curve. This is a bell-shaped curve as shown in the figure.



The Normal Curve.

Uses of the Normal Curve.

1. The normal curve is used for the comparison of an actual distribution with a normal (ideal) distribution. This is done by superimposing the normal curve upon the frequency polygon of the actual distribution.

2. The normal curve can be used to determine the probable reliability of the statistical results obtained from actual data.

3. The normal curve is used also to determine the percentage or proportion of the measures that should lie between any two selected points on the horizontal scale. For example, what percentage of the grades should be between 60 and 70, or between 90 and 100?

Skewness.

When the distribution is not symmetrical the curve will have a skewness either toward the right or toward the left. It means that the greatest concentration of the items is not in themiddle, but either to the right or to the left. Skewness means lack of symmetry.

D. Averages.

We have seen that a frequency table presents the facts classifed and condensed. The frequency graph gives a picture of the general trend of the facts. We still need to find a single number to represent the whole table. Such a representative (typical) number will give us a concise picture of the groups.and facilitate the comparison of different groups. For example, if we wish to compare the achievement of two classes in a test, we may read the grade of every student, but it is more suggestive of the real situation to state that the average of one class is 72 and of the other 80.

Wherever a single number is used to represent a group of numbers, that number is called a central tendency of the group. The central tendency gives the measure around which the measures largely cluster, usually in the central part of the series.

The most important measures of central tendency are the averages. The need for averages is seen by the frequency of their use in every day life. We speak of the average grade, average student, average wealth, average rainfall, average temperature, average crop, average wages, etc.

The averages most frequently used are the Mean, the Median, and the Mode.

I. The Mean.

The Mean is the good old arithmetic average. It is the result obtained by dividing the sum of all the items in the group by the number of items. Thus to find the mean grade of 100 students in mathematics we divide the sum of all the grades, 7267, by their number, which is 100. This gives 72.67, which is the mean.

The formula for the mean may be written

$$M = \frac{\Sigma X}{n}$$

in which M is the mean, ΣX (sigma X) is the sum of all the numbers in the series, and n is the number of items.

Finding the Mean from a Frequency Table.

Prepare a table with four columns, one for the class intervals, one for the midpoints, one for the frequencies, and one for the products of the midpoints and frequencies. To find the mean divide the sum of the last column by the total

number of frequencies. In finding the mean we assume (1) that the measures are distributed uniformly thorughout the interval, and (2) that they are represented by the value of the midpoint.

Intervals	m	f	f.m
30-39	35	2	70
40-49	45	3	135
50-59	55	11	605
60-69	65	20	1300
70-79	75	32	2400
80-89	85	25	2125
90-99	95	7	665
		n =100	7300

$$\text{Mean} = \frac{7300}{100}$$

Mean = 73
which is practically the same as the absolute mean of 72.67 which was computed by adding all the items.

Short Method of Computing the Mean.

When the table is very large the following method is used to redure the labor of multiplication:

1. Organize the data in a frequency table.
2. Assume the mean to be the midpoint of a class near the center of distribution (70-79).
3. Treating each class interval as a unit, arrange a column showing the deviation from the assumed mean of the midpoint of each interval with due regard to signs.
4. Multiply the deviation of each class by the frequency of that class and place the products in a new column (f×d).
5. Divide the algebraic sum of column f×d by the total number of frequencies. The result is the correction in class interval units.
6. Multiply this correction by the number of units in an interval.
7. Add the product algebraically to the assumed mean. The result is the required mean.

The following table illustrates the method. We shall assume 75 to be the mean.

Intervals	Mid class	f	d	f × d	
30-39	35	2	-4	-8	
40-49	45	3	-3	-9	
50-59	55	11	-2	-22	
60-69	65	20	-1	-20	
70-79	75	32	0	-59	
80-89	85	25	+1		25
90-99	95	7	+2		14
		100			+39

$$C = \frac{(-59 + 39)}{100} \times 10$$

C = -2
M = 75 + C
M = 75 - 2
Mean = 73

After calculating the mean, it is well to look at the X scale to see that the mean is reasonable, that is, that if falls somewhere in the central part of the range.

Properties of the Mean.

1. It is familiar to every one and the process of its computation is easy.

2. It gives weight to extreme deviations, and to all other deviations, in direct proportion to their size. This is sometimes a disadvantage because the occurrence of a few unusually large or unusually small items leads to a distortion.

3. Frequently it is not represented in the actual data as when the census gives the average family as 4.3 persons.

It is, therefore, desirable to have another measure of central tendency.

Exercise. Find the mean of the student heights.

II. The Median.

The median is the middle-sized item. If the measures are arranged in the order of size, the middle item is the median when the number of measures is odd. When the number of measures is even, the median is the number which is halfway between the two middle numbers.

Consider the series
10, 6, 5, 25, 15, 18, 20.

Arrange it in the order of magnitude
5, 6, 10, 15, 18, 20, 25.

The median is 15 because it is the middle number. The mean is 15.57.

Add another item to make the number even, thus, 5, 6, 10, 15, 18, 20, 25, 37.
The median is halfway between 15 and 18, that is 16.5.
The mean is 17.

The Median From a Frequency Table.

When the data is given in a frequency table the median is computed as follows:

Intervals	f
30-39	2
40-49	3
50-59	11
60-69	20
70-79	32
80-89	25
90-99	7
2	100
	50

$\frac{N}{2} = \frac{100}{2} = 50$

36 or 14 cases below the median

$Md = 70 + \frac{14}{32} \times 10 = 70 + 4\frac{3}{8} = 74\frac{3}{8}$

14 ←

18

32 or 18 cases below the median.

$Md = 80 - \frac{18}{32} \times 10 = 80 - 5\frac{5}{8} = 74\frac{3}{8}$

The method of finding the median is as follows:

1. Arrange the data in a frequency table.
2. Determine $\frac{n}{2}$. (in our example $\frac{n}{2} = 50$)
3. Begin at either end and add the frequencies in succession until the class containing the median is reached.
4. Divide the frequencies in this interval into two parts so that each side will have $\frac{n}{2}$ frequencies (14, 18).
5. The exact value of the median is then found by interpolation $\frac{14}{32} \times 10$ or $\frac{18}{32} \times 10$.

These steps are easily understood by examining the above example. The student will note that we assume uniform distribution of the items in each interval.

Properties of the Median.

1. It is not affected by items having extreme deviations from the normal, since it denotes rank.

2. It can be determined when the size of the extreme items is not known if the number of items is known.

3. The median is especially useful when the data are not capable of exact measurement as in measuring the intelligence of a group of students.

4. The median is not so adapted to mathematical computation as the mean. On the whole it is very valuable for practical use.

Exercise. Find the median of the student heights.

III. The Mode.

The Mode is the third measure of central tendency. It means the common thing, or the most usual occurrence. It is the most common height, the most common income, the most common grade. A frequency table is required to find the mode, but no calculation is necessary. In our example it is obviously 75.

Pearson's rule to find a value approximating the true mode is expressed by the formula

mode = mean - 3(mean - median)

That is, in moderately unsymmetrical distributions the median is always about 1/3 the distance from the mean towards the mode. In our example

$$\text{mode} = 73 - 3(73 - 74.375)$$
$$= 73 - 3(-1.375) = 77.1$$

Mean = 73; median = 74.38; mode = 77.1

When the curve is perfectly symmetrical, as in the normal curve, the mean, the median, and the mode are equal and fall at the same place in the diagram.

It is interesting to note that the mode is a point on the scale which has the highest ordinate in a polygon graph; that the ordinate drawn through the median divides the area under the curve into two equal parts; and that the mean is the abscissa of the center of gravity of the area under the curve.

This discussion has been designed to show that each type of average has its own particular field of usefulness. Each one is best for certain purposes and under certain conditions.

There are two other averages, the geometric mean and the harmonic mean but we cannot explain them in this limited discussion.

The following tables give the scores made in an intelligence test by the freshmen of four schools in The Pennsylvania State College 1925. Make a frequency table and a frequency graph for each. Find the mean, median, and mode for each school.

School A.

107	111	54	115	135	90	105	89	48	120	65
72	81	59	83	94	102	58	95	103	89	96
90	93	74	84	77	102	117	86	79	113	78
93	92	76	89	91	64	82	73	87	100	75
84	68	62	89	61	70	117	102	124	110	87
93	87	80	118	131	110	81	83	122	79	108
99	81	74	102	93	78	98	107	76	123	80
129	109	105	124	112	118	89	66	111	101	92

School B.

114	72	105	94	102	91	162	107	73	98	128
91	134	136	106	173	71	80	107	93	109	107
93	98	113	116	102	75	101	49	123	89	78
104	92	115	54	80	82	131	91	123	94	131
63	142	115	95	72	84	100	118	150	100	143
98	118	112	94	134	158	79	103	85	78	86
105	129	87	104	114	113	113	59	107	94	155
95	109	129	105	96	114	117	131	81	97	54

School C.

56	109	103	85	116	73	105	95	98	89	76	159
139	146	76	86	116	116	94	56	82	131	122	74
113	133	94	109	124	89	87	53	53	145	88	85
136	67	121	101	92	84	55	65	159	92	98	117
81	133	95	107	114	126	135	106	158	129	108	79
111	157	77	142	104	149	96	89	99	97	144	61
111	98	82	75	100	115	100	78	110	96	79	107
116	95	91	102	96	72	113	65	83	118	97	112
104	99	129	134	158	118	105	104	91	121	111	80
67	86	115	61	125	64	109	141	110	56	111	129
137	88	74	133	63	142	115	140	137	114	107	
97	105	104	169	100	82	118	96	132	90	68	

School D.

114	48	74	58	44	99	99	98	144	128	89	88
44	108	141	91	53	59	89	94	64	153	87	84
103	82	119	80	65	95	73	50	82	94	92	71
122	85	82	81	84	108	54	133	90	65	117	76
77	134	129	93	75	79	122	35	110	101	99	
72	54	92	109	98	35	72	77	66	114	115	
42	60	121	77	122	129	54	87	71	67	72	
123	49	119	67	73	78	71	64	73	82	44	
73	91	71	116	108	89	52	80	45	61	59	
127	57	66	51	126	74	88	98	56	124	70	
99	119	78	78	101	103	110	104	124	46	103	
96	77	144	104	96	78	64	136	79	62	114	

Chapter XIII.

S E R I E S

A series is a succession of numbers arranged in a definite order according to some law. In simple cases the law governing the formation of the series can be found.

Extend each of the following series to three more terms and state the law governing the formation of the series:

1.	1 , 2 , 3, 4 , ----	2.	14 , 11 , 8 , 5 , ----
3.	1 , 4 , 9,16 , ----	4.	1 , $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ----
5.	3 , 6,,12,24,,-----	6.	96 , 48 ,24 ,12 , ----
7.	1 , 3 , 5, 7 , ----	8.	2 , 4 , 6 , 8 , ----

In this chapter we shall study three kinds of series which are of greatest importance. These are the arithmetic series, the geometric series, and the binomial expansion. The arithmetic and geometric series are also called progressions.

I. The Arithmetic Progression.

An Arithmetic Progression is a series, in which each term is obtained by adding a given number to the preceding term. The number added each time is called the common difference.

Example. 2, 5, 8, 11, 14, ------

In an arithmetic progression, we have to consider five elements represented as follows:

a = first term d = common difference
L = last term n = number of terms
s = the sum of all the terms.

An arithmetic progression can be represented by
$a,\ a+d,\ a+2d,\ a+3d,\text{-----},\ L.$

Last Term.

If we wish to find the last term or any other term of a series it is not necessary to write all the terms. By inspection of the general series we find that the coefficient of d in any term is 1 less than the number of that term in the series. The formula for the last term is evidently

(1) $L = a+(n-1)d.$

This important formula involves four elements; if any three of which are known the fourth may be found.

Arithmetic Means.

The terms between any two terms of an arithmetic series are called the arithmetic means between those two terms.

Thus to insert three arithmetic means between 4 and 16 is to find the second, third, and fourth terms of the arithmetic series of five terms in which a = 4 and L = 16. Such problems are solved by substituting in formula (1) and finding the value of d.

Thus, $16 = 4 + (5 - 1)d$

$d = 3$

Therefore, the series is 4, 7, 10, 13, 16.

Sum of An Arithmetic Progression.

The sum of an arithmetic series can be found by simply adding all the terms together. It is not necessary, however, to do so many tiresome additions for we can easily derive a formula for the sum as follows.

Write the general series both forward and backward thus,

$$s = a + (a+d) + (a+2d) \text{ ---- } + (L - d) + L$$
$$s = L + (L - d) + (L-2d) \text{ ---- } + (a + d) + a$$

Adding, $2s = (a+L) + (a+L) + (a+L)$ ---- n times.

That is, $2s = n(a+L)$

(2) $$s = \frac{n}{2} \cdot (a+L)$$

Substituting the value of L from (1) we have,

$$s = \frac{n}{2} \left[2a + (n - 1)d\right].$$

Exercise.

1. Solve the formula $s = \frac{n}{2}(a+L)$ for L. Find the value of L when $s = 69$, $n = 6$ and $a = 4$.

2. In a contest there are eight prizes, $400, $375, $350, etc. What will the total amount be?

3. Find the sum of all the odd integers less than 100.

4. Write an A. P. of nine terms of which the first is 3 and the last is 19.

5. Insert one arithmetic mean between 3 and 81.
   Insert four means.

6. The bases of a trapezoid are 10 inches and 2 inches; what is the length of the median?

The general form of a geometric series is
a, ar, ar^2, ar^3 -------- L, where r is the common ratio.
The five elements of a geometric series are a, L, r, n and s.

Last Term.

From the general series it is evident that the exponent of r in any term is 1 less than the number of that term in the series. Hence the formula for the last term of a series of n terms is

$$(3) \qquad L = ar^{n-1}$$

Geometric Means.

The terms between any two terms of a geometric series are called geometric means between those two terms. If there is only one such term it is called the geometric mean (also mean proportional). To insert geometric means between two numbers we must first find the common ratio by using formula (3).

Insert three geometric means between 11 and 176.
11, —, —, —, 176.

Since $L = ar^{n-1}$, $176 = 11r^4$

$r^4 = 16$
$r = 2$, or -2.

The series is 11, 22, 44, 88, 176.
or 11,-22, 44,-88, 176.
Two more series with imaginary terms are possible.

Sum of a Geometric Progression.

To find a formula for the sum of a geometric progression we write

$$s = a + ar + ar^2 + ar^3 \text{ ---- } + ar^{n-1}.$$

Multiply by r, $rs = ar + ar^2 + ar^3 \text{ ---- } + ar^{n-1} + ar^n$.

Subtracting, $s - rs = a - ar^n$

$$(4) \qquad s = \frac{a - ar^n}{1 - r} \text{ or } \frac{ar^n - a}{r - 1}$$

Since $L = ar^{n-1}$, $rL = ar^n$, and

$$s = \frac{a - rL}{1 - r} \text{ or } \frac{rL - a}{r - 1}$$

IRREGULAR PAGINATION

Infinite Geometric Progression.

Formula (4) can be written $s = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$.

If r is less than 1, r^n will become smaller and smaller as n increases. Therefore, when $n \to$ infinity, $ar^n \to 0$, and

$s \to \frac{a}{1 - r}$. For brevity we say when $n = \infty$,

(5) $$s = \frac{a}{1 - r}.$$

Formula (5) is used only when the limit of the sum of an infinite number of terms is required and r is less than 1. If the ratio of a definite number of terms is required formula (4) is used whatever the value of r.

Example. Find the sum of the infinite series $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, -----

$$s = \frac{2}{1 - \frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4.$$

Exercise.

1. Find the fifth and tenth terms of 3, -6, 12, ----.

2. In a series of seven terms the last is 192 and the ratio is 2. What is the first term?

3. Test the series $\frac{1}{2}; \frac{1}{4}; \frac{1}{6}$ ----- to determine whether it is arithmetical or geometrical.

4. Insert one geometric mean between 3 and 81.
   Insert two geometric means between 3 and 81.

5. Insert one geometric mean between A and B.
   Prove that the geometric mean between two numbers is the square root of their product.

6. In the series $\frac{1}{3}$, 1, 3 ----, which term is 243?

7. Given a = 1, r = 3, s = 121; find L.

8. It is found by experiment that the number of bacteria doubles every three hours. What increase will there be in 12 hours, assuming that all outside conditions remain the same?

9. The population of a town is 10,000. It has increased at the rate of 5% of its population each year for the last ten years. What was the population ten years ago?

10. Find the amount of $1 at 5% compound interest for one year; for 2 years; for 3 years; for n years. Do these results form a geometric progression?

11. Find the sum of the infinite series $4, \frac{4}{3}, \frac{4}{9}$ --------.

12. Find the value of the repeating decimal .3333-----.

13. An old story about the invention of chess tells how the Indian king was so pleased with the game that he asked the inventor what reward he wished. The inventor suggested 1 grain of wheat for the first square, 2 grains for the second square, 4 for the third, 8 for the fourth, and so on for the 64 squares. Find an expression for the number of grains for which the inventor asked.

III. The Binomial Theorem.

By actual multiplication, we obtain the following results:

$$(a+b)^2 = a^2 + 2ab + b^2.$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

$$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

It is desirable that we have a method shorter than multiplication for finding a binomial expansion. Such a method was discovered by Isaac Newton and is called the Binomial Theorem.

To expand any power of a binomial like $(a+b)^n$:

1. The first term is a^n and the last is b^n.

2. The exponents of a decrease by 1, while the exponents of b increase by 1 in each term. The sum of the exponents of a and b in any term is n.

3. The coefficient of the first term is 1, and the coefficient of the second term is n.

4. If the coefficient of any term be multiplied by the exponent of a and the product be divided by the number of the term, the quotient is the coefficent of the next term.

5. The number of terms is n + 1.

6. The sign of each term is + if a and b are positive; the sign of each even term is - if b alone is negative.

The binomial theorem formula is

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2$$

$$\frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \text{--------} b^n.$$

in which the symbols 2! and 3! are called factorial 2 and factorial 3. They are also written $\lfloor 2$, $\lfloor 3$. They mean $2 \cdot 1$ and $3 \cdot 2 \cdot 1$ respectively.
$\lfloor 6 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Example. Expand $(2a - x)^5$

$$(2a - x)^5 = (2a)^5 - 5(2a)^4 x + 10(2a)^3x^2 - 10(2a)^2x^3 + 5(2a)x^4 - x^5$$

$$= 32a^5 - 80a^4x + 80a^3x^2 - 40a^2x^3 + 10ax^4 - x^5.$$

In expanding by the binomial theorem, the student should take the two steps as above, rather than try to write the result in one step.

Exercise.

Expand by the binomial theorem

1. $(a+2b)^4$ 2. $(2a - 3x)^5$ 3. $(x - \frac{1}{x})^6$

4. $(2a + \frac{1}{a})^5$ 5. $(a^2 - \frac{x}{2})^6$ 6. $(1 + .04)^5$

7. $(1 + .03)^4$ 8. $(1. - .05)^4$ 9. $(2x - .3)^5$

Write the first four terms in the expansion of:

10. $(1+x)^{\frac{1}{2}}$ 11. $(25+3)^{\frac{1}{2}}$ 12. $(8+4)^{1/3}$

13. $(1+x)^{-1}$ 14. $(1+x)^{2/3}$ 15. $(1 + .04)^{-3}$

Find the indicated roots to three decimal places:

16. $(11)^{\frac{1}{2}}$ 17. $(22)^{\frac{1}{2}}$ 18. $9^{1/3}$

Note. The binomial theorem is valid for fractional exponents, and for negative exponents when the first term is greater than the second.

Mendel's Law.

It is interesting to know that Mendel's Law of heredity follows the principle of the binomial theorem. The result of mixing W and B in the first generation is $(W+B)^2$, in the second generation $(W+B)^4$, in the third generation $(W+B)^8$, etc. The number of varieties is equal to the number of terms in the expansion. The numbers in the various kinds are proportional to the coefficients of the terms.

Two different kinds of chickens are allowed to mix freely. How many varieties would be produced in the second generation? the third? the fourth? What is the proportion in each generation?

Answer the same questions for three kinds of wheat planted side by side.

Chapter XIV
LOGARITHMS

Logarithms Save Labor.

In solving mathematical problems much time is consumed in performing arithmetical operations which are often long and laborious. To save time and labor many devices have been invented. Among these are tables of squares, adding machines, graphs, the slide rule, etc.

The most important of these labor-saving devices is the method of calculation by logarithms. By the use of logarithms,

1. Multiplication is reduced to addition.

2. Division is reduced to subtraction.

3. Raising to a power is reduced to one multiplication.

4. Extracting a root is reduced to a single division.

It is said that the method of logarithms has reduced the labor of computing to one tenth of what it was before. This section will explain the use of logarithms and their applications.

Numbers as Powers of 10.

The idea back of logarithms is that every positive number may be expressed exactly or approximately as a power of 10. Hence to make a calculation we have merely to combine exponents. For example, we may multiply 100 by 1000 by first expressing them as powers of 10 and adding the exponents. Thus

$$100 \times 1000 = 10^2 \times 10^3 = 10^5 = 100{,}000.$$

In the same manner we may multiply 25.4 by 5.6 by adding the exponents of the powers of 10, which equal 25.4 and 5.6 respectively, and then finding the number which corresponds to the new power of 10. This raises two questions:

(1) What powers of 10 equal 25.4 and 5.6?

(2) What is the value of 10 when raised to the sum of these two powers?

These questions may be answered by stating that mathematicians have computed the various exponents of 10 and arranged them in tables for convenient reference.

From the tables we find that

$$25.4 = 10^{1.40483}$$

$$5.6 = 10^{0.74819}$$

$$\text{Product} = 10^{2.15302} \quad \text{(adding the exponents)}$$

Then by referring to the table we see that the number corresponding to this new power of 10 is 142.2+.

$$\therefore \quad 25.4 \times 5.6 = 142.2+.$$

A Logarithm Is an Exponent.

In the equation $25.4 = 10^{1.40483}$ the exponent 1.40483 indicates the power to which 10 must be raised to produce 25.4 This exponent is called the logarithm of 25.4 to the base 10. Thus we have the

Definition. The logarithm of a number to the base 10 is the exponent to which 10 must be raised to produce that number.

The student will do well to remember that a logarithm is an exponent. Thus

$10^3 = 1000$ is the same as $3 = \log_{10} 1000$.

That is, to say 3 is the exponent to which 10 must be raised to equal 1000 means the same thing as saying, 3 is the logarithm of 1000 to the base 10, written 3 = log 1000, the base 10 being understood.

Characteristic and Mantissa of a Logarithm.

Consider the following table of powers of 10:

Exponential form	Logarithmic form
$10^3 = 1000$	$\log 1000 = 3$
$10^2 = 100$	$\log 100 = 2$
$10^1 = 10$	$\log 10 = 1$
$10^0 = 1$	$\log 1 = 0$
$10^{-1} = .1$	$\log 0.1 = -1$ or $9-10$
$10^{-2} = .01$	$\log 0.01 = -2$ or $8-10$
$10^{-3} = .001$	$\log 0.001 = -3$ or $7-10$

It is obvious from the table that the logarithm of a number between 100 and 1000 must lie between 2 and 3; that is, it equals 2 + a decimal. The logarithm of a number between 10 and 100 is 1 + a decimal. The logarithm of a number between 1 and 10 is 0 + a decimal. The logarithm of a number between 0.1 and 1 is -1 + a decimal. The logarithm of a number between 0.01 and 0.1 is -2 + a decimal, and so on.

Thus every logarithm has two parts, an integer and a decimal fraction. The integral part is called the Characteristic. The decimal part is called the Mantissa. The logarithm is the characteristic plus the mantissa. Hence to find the logarithm of any number we must find both the characteristic and the mantissa.

Note. The mantissa of an integral power of 10 is zero. Thus, log 100 = 2.00000. Observe also that the characteristic of a number between 1 and 10 is zero. Care must be taken not to omit the characteristic even when it is zero.

Finding the Characteristic.

The characteristic of a number is easily determined by inspection. Its value depends merely upon the position of the decimal point.

It is clear from our discussion of the table of powers of 10 that the characteristic of a number which contains one digit left of the decimal point (as 3.7 or 8.24, or any number between 1 and 10) is 0; the characteristic of a number having 2 places left of the decimal point (as 54 or 80.32 or any number between 10 and 100) is 1; that of a number having 3 integral digits is 2. In general it will be seen that the following law is true:

The First Law of the Characteristic. When the number is greater than 1, the characteristic is positive, and is one less than the number of digits left of the decimal point.

From the same table it is also seen that the characteristic of a number between 0.1 and 1 (such as 0.4 or 0.735) is -1; that of a number between 0.01 and 0.1 (as 0.05 or 0.0693 or any pure decimal with only one zero immediately at the right of the decimal point) is -2; that of a number between 0.001 and 0.01(as 0.006 or 0.00587 or any pure decimal with only two zeros immediately at the right of the decimal point) is -3. In general, it will be seen that the following law is true:

The Second Law of the Characteristic. When the number is less than 1(that is, a pure decimal between 0 and 1) the characteristic is negative, and is one more than the number of zeros immediately to the right of the decimal point.

log 254 = 2+a decimal log 0.254 = -1+a decimal

log 25.4 = 1+a decimal log 0.0254 = -2+a decimal

log 2.54 = 0+a decimal log 0.00254 = -3+a decimal

Note. The logarithm of a negative number is an imaginary number, and is not used in computation.

Finding the Mantissa

The mantissa or decimal part of the logarithm of a number is found from the tables. These tables are called tables of logarithms, but they are really tables of mantissas because they have nothing to do with characteristics (since these are determined by inspection).

(1) If $10^{0.40483} = 2.54$, or log 2.54 = 0.40483

multiplying both sides of equation (1) by 10, we get

$10^{1.40483} = 25.4$, or log 25.4 = 1.40483

again multiplying by 10, we get

$10^{2.40483} = 254$, or log 254 = 2.40483, and so on.

Observe that the position of the decimal point in the number affects the characteristic only, but the mantissa remains the same.

In like manner, if we divide both sides of equation (1) by 10 successively, we get

$10^{0.40483 - 1} = 0.254$, or log 0.254 = 0.40483 - 1

$10^{0.40483 - 2} = 0.0254$, or log 0.0254 = 0.40483 - 2

and so on. Thus the mantissa of 254 is the same as the mantissa of 25.4, of 2.54, of .254, of 0.0254, etc. The logarithms of these numbers differ only in their characteristics. This establishes the following:

Law of the Mantissa. The mantissa is the same for all numbers having the same succession of digits whatever may be the position of the decimal point.

From this it follows that in looking up the mantissa in the tables do not pay any attention to the decimal point, since the position of the decimal point affects the characteristic only.

The two laws of the characteristics and the law of the mantissa are true because 10 is used as the base.

TIGHT BINDING

The Mantissa Is Positive.

If in log 0.0254 = -2+.40483 we combine the negative characteristic and the positive mantissa, we get log 0.0254-= -1.59517. But this changes the form of the mantissa for the same sequence of figures and is inconvenient for computation. To avoid this and to secure the advantage of having the same mantissa for the same order of figures, we agree to express the logarithm of any number in such a form that its mantissa shall be positive.

If log 0.0254 = - 2 + .40483, we cannot write it -2.40483, for this would mean that both characteristic and mantissa are negative, while the mantissa is positive. Some books write it $\bar{2}$.40483(the dash above the 2 means that only the characteristic is negative). Others write it 0.40483 - 2. The most convenient form, however, is 8.40483 - 10.

Using the Tables. (1) To find the logarithm of a number.

In finding the logarithm of a number it is best to determine the characteristic first, and then look up the mantissa in the table. A part of a typical page of five-place tables is reproduced here. The mantissas appear in the body of the table. They are decimals of five places throughout. (The first two figures appear in the first column only). Notice that when advancing in the tables, the mantissas increase.

N	0	1	2	3	4	5	6	7	8	9
250	39794	811	929	846	863	881	898	915	933	950
51	967	985	*002	*019	*037	*054	*071	*088	*106	*123
52	40140	157	175	192	209	226	243	261	278	295
53	312	329	346	364	381	398	415	432	449	466
54	483	500	518	535	552	569	586	603	620	637
55	654	671	688	705	722	739	756	773	790	807

If the number consists of four figures, the first three are shown in the first column which has the letter N at the top. The fourth figure is found at the top of one of the columns marked 0, 1, 2, 3,--- 9.

The mantissa sought is in the row of the first three figures and in the column of the fourth figure.

If the number consists of one, two, or three figures, find the number in the N column and the mantissa at the right in the zero column.

If the number consists of five or more figures, its mantissa will be roughly equal to the mantissa of the nearest number of four figures.

Examples. $\log 253.4 = 2.403^{81}$ $\log 0.25436 = 9.40552 - 10$

$\log 25 = 1.39794$ $\log 2.516 = 0.40071$

(2) To Find the Number Corresponding to a Logarithm.

If $\log 25.4 = 1.40483$, then 25.4 is the number corresponding to the logarithm 1.40483. It is sometimes called the Anti-logarithm.

When the logarithm of a number is given, the number is found by reversing the process of finding the logarithm when the number is given.

Example.

Find the number whose logarithm is 1.40586.

The characteristic 1 tells us only the position of the decimal point. To find the figures of the number we must look up the mantissa .40586 in the body of the table. To the left of this mantissa in column N, we find the first three figures of the number, 254, and at the head of the column of the mantissa we find the fourth figure of the number, 6. The number then has the figures 2546. We do not know the position of the decimal point until we consider the characteristic. Since the characteristic is 1, we must have two figures left of the decimal point.

$$\therefore \quad 1.40^{5}86 = \log 25.46$$

A change in the characteristic changes only the position of the decimal point. Thus,

$4.40486 = \log 25460$, and $7.40486 - 10 = \log 0.002546$.

If the decimal part cannot be found exactly in the table, find the number corresponding to the nearest decimal. Thus,

$$1.40375(1.40381) = \log 25.3^{4}.$$

We can get a closer approximation by interpolation.

Interpolation.

If the number whose logarithm is to be found contains more than four digits, its logarithm cannot be read directly from the table. But it may be found by interpolation.

Example. Find the logarithm of 254.76

10	6	log 254.70	= 2.40603	Tab. Difference
		log 254.76	= ?	= 17
		log 254.80	= 2.40620	

Since 254.76 is between 254.70 and 254.80, its logarithm must be between their logarithms. And since 254.76 is $\frac{6}{10}$ of the way from 254.70 to 254.80, we add .6 of the tabular difference 17 to the logarithm of 254.70. Thus,

log 254.76 = 2.40603 + .6 of tabular difference.
.6 x 17 = 10.2 (hence we add 10)

$\therefore$ log 254.76 = 2.40613.

Observe that in using five-place tables it is sufficient to obtain results to the nearest figure in the fifth place.

It remains to show how to find by interpolation a number whose mantissa is not found exactly in the table. In this case two other consecutive mantissas can always be found between which the given mantissa lies. The number of four figures corresponding to the smaller of these mantissas gives the first four figures of the number sought. The fifth, and even the sixth figure, can then be found by interpolation.

Example.

If log N = 0.40255, find N.

10	log 2.5260	= 0.40243	12	Tab. difference
	log N	= 0.40255		
	log 2.5270	= 0.40261		= 18

The number we want lies between 2.5260 and 2.5270. Now since an increase of 18 in the mantissa produces an increase of 10 in the number, an increase of 12 in the mantissa must produce an increase of $\frac{12}{18}$ of 10 (= 6.6 or 7) in the number. Hence N = 2.5267.

In the process of interpolation we assume that the increase in the logarithm is directly proportional to the increase in the number. This assumption is not strictly true, but the results obtained are nearly always correct to the fifth place.

With a little practice, interpolation could be easily carried out mentally or by the use of proportional parts. The student should train himself with that end in view. Ability to interpolate in one table will help one to interpolate in other tables.

Exercise.

1. Verify the following:

(a) log 2.805 = 0.44793
(b) log 0.6028 = 9.78017 - 10.
(c) log 7 = 0.84510.
(d) log 0.084 = 8.92428 - 10.

2. Verify the following:

(a) log 3.1416 = 0.49715.
(b) log 2.7183 = 0.43429.
(c) log 587.69 = 2.76915.
(d) log 0.37742 = 9.57682 - 10.

3. Find the logarithms of the following numbers:

(a) 763.4 ; 5.346 ; 2 ; 300 ; .01.

(b) 0.4077; 6313.4 ; 2.4386; 0.014165.

4. Find the numbers whose logarithms are:

(a) 1.43233 ; 9.60206-10 ; 0.73046.

(b) 2.07390 ; 0.58122 ; 8.63025-10.

5. Express these numbers as powers of 10.

854.3 ; 572400 ; 600 ; 0.001

6. Express in the language of logarithms.

$142 = 10^{1.15229}$; $6 = 10^{0.77815}$

$0.3281 = 10^{9.51601-10}$

7. Write as powers of 10 the numbers whose logarithms are:

2.69020 ; 0.74472 ; 7.97067-10.

8. What is the meaning of

$x^{\frac{1}{2}}$; $x^{\frac{3}{4}}$; $x^{\frac{13}{10}}$; $x^{1.562}$

9. What is the value of

(a) $10^{\log 100}$ (b) $10^{\log 2.314}$ (c) $\log 10^{3}$

(d) $\log 10^{1.72493}$

10. Find the values of the following by first expressing each number as a power of 10 and then combining the exponents. Finish by finding the values of the new powers of 10.

(a) 4.321×0.62 (b) 1760×10.82

(c) $531.4 \div 73.8$ (d) $9.092 \div 0.8146$

(e) $(61.4)^2 \times 2.22$ (f) $\sqrt{5280} \div 48.07$

(g) $\dfrac{\sqrt{104.6} \times 2.536}{5.83}$ (h) $\dfrac{(6.314)^2 \times 0.0714}{203.3}$

The Laws of Exponents.

Since logarithms are exponents, the laws of exponents must apply also to logarithms. These laws are familiar to the student from algebra. They are:

I. $a^m \cdot a^n = a^{m+n}$ II. $\dfrac{a^m}{a^n} = a^{m-n}$

III. $\left(a^m\right)^n = a^{mn}$ IV. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

The Laws of Logarithms.

The laws of exponents may be restated in logarithmic form as follows:

I. Logarithm of a Product. The logarithm of a product is equal to the sum of the logarithms of its factors.

For if $10^m = x$ and $10^n = y$,

Then, $m = \log x$ and $n = \log y$ (by definition)

Multiplying, $xy = 10^m \cdot 10^n = 10^{m+n}$

Therefore, $\log xy = m+n$

Formula I. $\log xy = \log x + \log y$

Example. $\log 15 = \log 3 + \log 5$.

II. Logarithm of a Quotient. The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

For if $10^m = x$ and $10^n = y$

Dividing, $\frac{x}{y} = 10^m \div 10^n = 10^{m-n}$

Therefore, $\log \frac{x}{y} = m - n$

Formula II. $\log \frac{x}{y} = \log x - \log y.$

Example. $\log \frac{5}{7} = \log 5 - \log 7.$

III. Logarithm of a Power. The logarithm of a power of a number is equal to the exponent times the logarithm of the number.

For if, $x = 10^m$, then $m = \log x.$

Therefore, $x^n = 10^{mn}$

Therefore, $\log x^n = mn = n \log x$

Formula III. $\log x^n = n \log x$

Example. $\log(13.5)^2 = 2 \log 13.5.$

IV. The Logarithm of a Root. The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.

For if, $x = 10^m$, then $m = \log x.$

Taking n^{th} root, $\log x^{\frac{1}{n}} = 10^{\frac{m}{n}}$

Therefore, $\log x^{\frac{1}{n}} = \frac{m}{n} = \frac{\log x}{n}$

Formula IV. $\log \sqrt[n]{x} = \frac{1}{n} \log x.$

Example. Find the cube root of 54,7

$$\log \sqrt[3]{54.7} = \frac{1}{3} \log 54.7 = \frac{1}{3}(1.73799) = 0.57933.$$

Hence, $\sqrt[3]{54.7} = 3.796.$

Remark. The above laws may be reversed as follows:

$$\log 3 - \log 4 = \log \tfrac{3}{4}$$

$$\tfrac{1}{2} \log A + 3 \log B = \log A^{\frac{1}{2}} + \log B^3 = \log A^{\frac{1}{2}} \cdot B^3.$$

Exercise.

Expand the following expressions by using the laws of logarithms:

1. $\log \frac{15 \times 47.3}{2}$. 2. $\log \frac{\sqrt{xy}}{a}$. 3. $\log \frac{x^2}{\sqrt{y}}$.

4. $\log \sqrt{67^2 - 43^2}$. 5. $\log \frac{n-1}{n+1}$. 6. $\log(x^2 - x - 2)$

Express in one term.

7. $\log 6 + \frac{1}{2}\log 5 - \log 11$. 8. $2 \log x - \frac{1}{3} \log y$.

9. $\log(x^2 - 1) - \log(x+1)$. 10. $\frac{1}{2}\log(c - b) + \frac{1}{2}\log(c+b)$

11. Is $\log 18 = \log 2 \times \log 9$? Why?

12. Is $\log\left(\frac{7}{5}\right) = \frac{\log 7}{\log 5}$? Why?

13. Is $\frac{\log 3}{\log 7} = \frac{3}{7}$? Why?

14. Show that $10^{\log n+1} = 10n$.

Computation by Means of Logarithms.

The application of logarithms to shorten the work of computation depends upon the four laws of logarithms given above. By using these laws, multiplication is replaced by addition; division by subtraction; raising to a power by a multiplication; and extracting a root by a division. Logarithms cannot assist in the operations of addition and subtraction.

Arrangement of Work.

Before looking up any of the logarithms in the table, we should write out a skeleton for all the work. This form should provide places for the logarithms as they are found in the table, and places for the modified logarithms. This plan will save time in the end and keep the work in a presentable form. Logarithms are used where they are really a help, not for the mere sake of using them.

Example. Find the value of

$$\frac{0.0843 \times (91.34)^{\frac{2}{3}}}{(7.2)^2 \times \sqrt[3]{0.365}} = x$$

Plan: $\log x = \boxed{\log 0.0843 + \frac{2}{3} \log 91.34} - \boxed{2 \log 7.2 + \frac{1}{3} \log 0.365}$

$\log 0.0843$	=	8.92583 - 10
$\frac{2}{3} \log 91.34$	= $\frac{2}{3}(1.96066)$ =	1.30711
$\log N$	=	0.23294
$2 \log 7.2$	= $2(0.85733)$ =	1.71466
$\frac{1}{3} \log 0.365$	= $\frac{1}{3}(29.56229 - 30)$ =	9.85410 - 10
$\log D$	=	1.56876
$\log N$	= 10.23294 - 10	
$\log D$	= 1.56876	
$\log X$	= 8.66418 - 10	
X	= 0.04615 Ans.	

Note 1. Logarithm of a Fraction.

To find the logarithm of a fraction, add the logarithms of the factors of the numerator, and from the result subtract the sum of the logarithms of the factors of the denominator. (Why?)

Note. 2. Avoiding Fractional Characteristics.

To divide a negative logarithm by a number, write the logarithm in such a form that the negative part of the characteristic may be exactly divisible by that number and give -10 as a quotient. Thus in taking $\frac{1}{3} \log 0.365 = \frac{1}{3}(9.56229 - 10)$, we added 20 and subtracted 20 to get it in the form $\frac{1}{3}(29.56229 - 30)$.

After dividing by 3, the result is 9.85410 - 10. If the divisor had been 2, we would have written it $\frac{1}{2}(19.56229 - 20)$.

Note 3. Avoiding Negative Mantissas.

Since the tables give only positive mantissas, we take care to keep the mantissas positive at every step of the calculation. When a larger logarithm is to be subtracted from a smaller, we add 10 and subtract 10 from the smaller logarithm. Thus, in the example above we had to subtract log D = 1.56867 from log N = 0.23294; so we wrote log N = 10.23294 - 10 then we subtracted and got a positive mantissa.

Calculations with Negative Numbers.

We have seen that a negative number has an imaginary logarithm which cannot be used in calculation. But since the numerical results of multiplications and divisions are the same no matter what the algebraic signs are, we can make calculations involving negative numbers as follows: First, decide whether the final result should be positive or negative. Then find the numerical value of the expression by logarithms, treating all numbers as positive. Finally give the result the proper sign.

Example. Find the value of $\sqrt[3]{\frac{-2.78}{(-1.53)} \times \frac{(-46.2)^2}{(-8.8)^5}}$

The combined effect of these signs gives a negative result. So we write

$$-x = -\frac{\sqrt[3]{2.78\ (46.2)^2}}{1.53(8.8)^5}$$

$$\log x = \left[\frac{1}{3}\log 2.78 + 2\log 46.2\right] - \left[\log 1.53 + 5\log 8.8\right].$$

Complete the calculation, and give the result a negative sign.

Accuracy.

Since logarithms in general are irrational numbers, their value cannot be found exactly. But it can be found approximately to four, five, or any desired number of decimal places. For this reason, logarithmic computation gives only approximate results. The results are not in any case correct beyond the number of decimal places in the logarithms used. A five-place table is sufficient for nearly all engineering problems since the most precise measurements do not exceed this degree of accuracy. In computing problems of finance and problems of astronomy, seven-place tables or even ten-place tables are sometimes used.

Exercise.

Find the values of the following:

1. $\sqrt{73.6}$ 2. $\sqrt[3]{73.6}$ 3. $\sqrt[3]{(6.345)^2}$.

4. $(0.03516)^{\frac{3}{2}}$. 5. $(0.001)^{\frac{3}{4}}$. 6. $\sqrt[3]{-.1306}$.

7. $\frac{4}{3} \times 3.1416 \times 15^3$. 8. $\pi\,(2.64)^2(.412)$.

9. $\frac{1}{3}\pi(18.3)^2\,20.6$. 10. $2\pi(87)^2(\ .7 + 15.3)$

11. $\dfrac{837.1 \times 2.782}{7309}$. 12. $\dfrac{1.375 \times 0.06423}{38.21}$.

13. $\dfrac{0.281 \times 207 \times .00196}{50.8 \times 3.177}$. 14. $\dfrac{12.8 \times 0.369 \times 0.0535}{265 \times 1.08}$

15. $\dfrac{(44)^2 \times (1.11)^{\frac{1}{5}}}{\sqrt[3]{1000}}$. 16. $\dfrac{(7.3)^3 \times 0.02054}{\sqrt{245050}}$.

17. $\dfrac{\sqrt[3]{0.512}\ \sqrt[4]{3.075}}{\sqrt{80}}$. 18. $\sqrt[3]{\dfrac{1.57(-28.4)^2}{(17.5)^3\sqrt[3]{100}}}$.

19. $\dfrac{\sqrt{.1303 \times (13.953)^{\frac{5}{3}}}}{\sqrt{(4.1)^3(22.594)\sqrt[3]{0.0973}}}$. 20. $\dfrac{\log 0.7326}{40.5}$.

21. $\dfrac{1.85\sqrt{18.14}\ (-2.367)^2}{(0.4263)^3 \times \sqrt[3]{-927.86}}$ 22. $\dfrac{\log 3.7026}{\log 0.08065}$.

23. $K = \sqrt{s(s-a)(s-b)(s-c)}$. Find K when a = 76.3, b = 105.6, c = 90, and $s = \frac{1}{2}(a+b+c)$.

24. $s = \dfrac{m\,g\,l}{\pi\,r^2\,k}$. Find k when m = 867.4, g = 980, l = 225.3, r = 0.35, and s = 0.071.

25. $t = \pi\sqrt{\dfrac{l}{32.2}}$. Find l when t = 2.

26. The radius of the earth is 3960 miles. Find its circumference, its surface, and its volume.

27. How many cubic feet of silage will a silo hold, which is 43 ft. 6 in. high, if the radius of the base is 6 ft. 4½ inches?

28. Sirius, the brightest star, is distant from the earth 5.26×10^{13} miles. If it were blotted out at this instant how long would it be before we find it out? (Light travels 186000 miles a second).

29. The best elevation for the outer rail of a railway curve is given by the formula

$$E = \frac{12\ G\ V^2}{32\ .\ 2\ R}.$$ Calculate E if G = 4.625,

V = 72, and R = 5280.

30. The radius of the circle inscribed in a triangle is given by the formula:

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$ Find r if a = 15.2,

b = 11.08, c = 8.3, and $s = \frac{1}{2}(a+b+c)$.

Cologarithms.

It is possible to perform any computation involving multiplication, division, powers, and roots, by a single addition. This is done by using cologarithms.

The Cologarithm of a number is the logarithm of the reciprocal of the number.

$$\text{Colog } N = \log \frac{1}{N} = \log 1 - \log N = 0 - \log N = -\log N.$$

Thus the logarithm of a number equals minus the logarithm of the number.

Cologarithms, like logarithms, are written with positive mantissas. Consequently, to find the cologarithm of a number, subtract the logarithm of the number from 0, written 10.00000 - 10. This is done mentally by looking at the logarithm of the number (in the table) and subtracting in order each figure of the logarithm (including the characteristic) from 9, except the last figure which must be subtracted from 10.

Thus colog 45.26 = log 1 - log 45.26

log 1 = 0 = 10.00000 - 10

log 45.26 = 1.65571

colog 45.26 = 8.34429 - 10

colog 0.04526 = 10 -(8.65571 - 10) - 10 = 1.34429.

Since $\log \frac{M}{N} = \log M - \log N = \log M + \text{colog } N$, instead of subtracting the logarithm of a number, we may add its cologarithm. The advantage of cologarithms is that when more than two logarithms are to be combined by addition and subtraction, the cologarithms enable us to complete the operation by a

single addition. There is no gain, however, im using cologarithms when we have a quotient of only two numbers. To beginners cologarithms are a source of errors. Students should not use them until they have thoroughly mastered the method of logarithms.

Exponential Equations.

Compare the equations (1) $x^3 = 8$ (2) $2^y = 8$. These equations are of entirely different kinds. Equation (1) asks the question: What number cubed equals 8? (or, what is the cube root of 8?) Equation (2) asks the question: To what power must 2 be raised to obtain 8? The first has a constant exponent, and is called an algebraic equation (a power function). The second has an unknown in the exponent , and is called an Exponential Equation.

Some exponential equations can be solved by inspection. The value of x in equation (2) above is obviously 3. If $3^y = 81$, the value of $x = 4$. But suppose we have the equation $3^y = 24$, To what power must 3 be raised to obtain 24? We cannot find the value of x by inspection. How shall we find it?

Since 3^y and 24 are equal, their logarithms are equal. Taking the logarithms of both sides, we have

$$\log 3^y = \log 24$$

$$x \log 3 = \log 24$$

$$x = \frac{\log 24}{\log 3} = \frac{1.38021}{0.47712} = 2.8928.$$

Remember that $\frac{\log 24}{\log 3}$ is not equal to $\log \frac{24}{3}$. The first calls for the division of a logarithm by another. The second indicates the logarithm of a fraction and involves subtraction.

Other Bases.

The logarithms which we have been using are based on the fact that every number is some power of 10. But it is also true that every number is some power of 2, 3, 4, 5, or of any other positive number except 1. It is, therefore, possible to have other systems of logarithms based upon powers of 2, 3, 25, 100, etc.

For example: $2^4 = 16$. The exponent 4 is the logarithm of 16 to the base 2. If $7 = 3^{1.7713}$, the exponent 1.7713 is the logarithm of 7 to the base 3, written $\log_3 7 = 1.7713$.

General Definition of Logarithm.

The logarithm of any number to any base is the exponent of the power to which the base must be raised to give the number.

The logarithms to the base 10 are called the Common Logarithms because they are most commonly used. The fact that moving the decimal point in a number does not change the mantissa but merely adds or subtracts some integer from the characteristic, is true only of logarithms to the base 10, and makes common logarithms the best for numerical computations.

Natural Logarithms.

Only one other base is generally used, and that is the base e = 2.71828----, called the natural base. This irrational number is chosen because it is found in the calculus that when the rate of change in a function is at any point equal to the value of the function itself at that point, the function is given by the equation

$$y = e^{x}, \text{ where}$$

$$e = 1 + \frac{1}{1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} \text{ ----} = 2.71828 \text{ ----}$$

Hence e, like π, is an important constant of nature, fundamental in advanced mathematics. The natural logarithms are called Napierian logarithms after their discoverer John Napier(1614). The adoption of 10 as a base is due to Henry Briggs (1617).

General Laws.

No matter what base is used the four laws of logarithms for multiplication, division, raising to a power and finding a root apply.

For any base b it is true that

(1) $\log 1 = 0$, since $b^0 = 1$

(2) $\log b = 1$, since $b^1 = b$

(3) $\log 0 = -\infty$, (when $b > 1$)

Change of Base.

To find the logarithm of any positive number to any base such as log 7 to the base 3, we may write

$$\log_3 7 = x, \text{ hence } 7 = 3^x.$$

Taking logarithms to base 10, $\log 7 = x \log 3$

or $x\ (= \log_3 7) = \frac{\log 7}{\log 3} = \frac{.84510}{.47712} = 1.77713.$

More generally, $\log_b N = \frac{\log_a N}{\log_a b}$. (1)

If a is put for N in (1), we get

$$\log_b a = \frac{1}{\log_a b}, \text{ or } \log_b a \cdot \log_a b = 1 \quad (2)$$

Whence, $\log_e 10 \cdot \log_{10} e = 1.$

It follows from formula (1) that if we know the logarithms of all numbers to some base a, the logarithms of all numbers to any other base b may be found by dividing all the logarithms of the first system by one and the same divisor, namely by the logarithm of the second base with respect to the first.

Exercise.

1. Write a brief sketch of the discovery of logarithms by Napier and Briggs.

2. Solve problems 12, 14, and 16 of the preceding exercise by cologarithms.

3. Show that

   (a) $\log_{10} e = 0.4343$ (b) $\log_e 10 = 2.303$

   (c) $\log_e 426 = 6.05444.$

   (d) $\log_e 0.0763 = - 2.57309.$

Solve the following equations:

4. $2^x = 64.$ 5. $4^x = 100.$ 6. $5^x = 10.$

7. $(1.12)^x = 6.$ 8. $2^{x-4} = 5.$ 9. $e^x = 2.$

10. $(1.06)^x = 1.45.$ 11. $12^{2x+5} = 15^{x+3}$ 12. $e^x = e^{-x}$

13. $\log_{10} x \cdot \log_e x = \log_{10} x^3.$ 14. $\log_e x^2 - \log_e x = 7.$

15. Find the logarithms of
    (a) 75 to the base 4. (b) 25 to the base 3.

16. How would you make a table of logarithms to the base 7?

Chapter XV.

THE MATHEMATICS OF FINANCE

In this chapter we shall study the application of mathematics to certain financial problems. This should give the student an appreciation of exact science in business.

I. Simple Interest.

Interest.

Interest is a consideration for the use of capital. The invested capital is called principal; the sum of the principal plus the interest is called the amount. The rate of interest is the interest earned on a unit of principal in a unit of time. Thus r stands for the interest on 1 dollar for 1 year although it is ordinarily expressed as a rate per hundred instead of a rate per unit.

Simple interest is computed on the original principal only, and is proportional to the rate and the time.

The fundamental formulas of simple interest are:

(1) $i = P r t$

(2) $A = P(1 + rt)$ (Since Amount = P + i)

Each of these formulas involves four variables. If any three of these are known, the fourth can be easily found.

Ordinary and Exact Interest.

If simple interest is calculated on the basis of 360 days in a year; it is called ordinary interest. If it is calculated on the basis of 365 days in a year, it is called exact interest.

When the rate is 6% and the time is 60 days, business men compute ordinary interest by moving the decimal point in the principal two places to the left. (Why?). Thus the interest on $542.83 at 6% for 60 days is $5.43.

Exercise.

1. Solve the interest formulas for each letter in terms of the others.
2. Find the ordinary interest on $2778.40 for 83 days at 4.5%.

3. Find the exact interest at 5% on $4289 for 96 days.

4. Find the ordinary interest on a note for $720 dated March 12, 1926 and paid June 7, 1926 with interest at 8%.

5. Find the ordinary and exact interest on $3500 at 5% from April 15, 1926 to October 25, 1926. Which method is more favorable to the debtor?

6. What sum of money placed at interest for $1\frac{1}{2}$ years at 5% amounts to $826?

7. In how many years will $240 at $5\frac{1}{2}$% gain $45.

8. At what per cent will $1220 gain $180 in 4 years?

9. In how many years will a dollar double itself at 6%?

10. At what per cent will $750 amount to $1000 in 6 years?

11. A merchant is offered a $25 discount on a $600 bill due after 60 days. If he pays cash, at what rate would his money be earning interest for the next sixty days?

12. Which would you rather pay $98.00 after 30 days or $100 after 90 days?

13. Write formulas for exact interest and ordinary interest and find their rate.

Present Value.

It is sometimes necessary to find the value now of a sum of money due at some future time. The value now of a sum of money A due n years from now is called the present value, or present worth, of A. Present value is that sum which, if invested today at a given rate, will amount to the given sum on the given future date. It is obvious that in formula (2) P represents the present value, while A represents the sum due at some future time. Solving (2) for P we have

$$P = \frac{A}{1 + rt}$$

which is the expression for the present value at simple interest.

Simple Discount.

Discount is a consideration for the payment of a sum of money before it is due. We have seen that interest is computed on the principal invested, P, and is paid at the end of the interest period. Discount is computed on the amount specified in the face of the note or bill and is deducted or paid in advance.

The discount on A is the difference between A and its present value. Thus

$$D = A - P \quad \text{or} \quad P = A - D$$

Let d denote the rate of discount. Then the discount on A due at the end of t years is

$$(3) \qquad D = A\,dt$$

From $P = A - D$, we have $P = A - Adt$ or

$$(4) \qquad P = A(1 - dt)$$

Bank Discount.

If a farmer asks for $100 loan for a year from a bank which is charging 7% discount, the bank will make the farmer sign a note promising to pay $100 at the end of a year. Then the bank will give the farmer 100- 7 = $93. This bank discount is commonly referred to as interest paid in advance.

It must be noted that the interest rate which the farmer is paying is more than 7%, but the banker cannot be accused of dishonesty for he quoted 7% discount, not 7% interest. Bankers have adopted this method of computation partly because it is easier to find P when the discount rate is given than when the interest rate is known.

Exercise.

1. A note for $312.50 is due in 3 months. What is its cash value if the rate of discount is 8%?

$$P = 312.50\left(1 - \frac{.08}{4}\right)$$

$$= 312.50(.98) = \$306.25.$$

2. Find the discount rate if $690 is the present value of $700 due after 60 days.

3. A man presented to his bank for discount a note which would mature 66 days later, the face of the note being $700 and without interest. The bank discounted the note at 6%. How much did the man realize upon it?

4. Find the principal which would amount to $700 in 66 days at 6% interest.

5. Find the present value of $500 due in 120 days, money being worth 6%.

6. Find the present value of $2000 due after four months if money is worth 9%.

7. Find the discount rate if the discount on $500 due after one year is $27.50.

8. A borrowed from B $720 and agreed to repay it after eight months with interest at the rate of 5.25%. How much did A pay at the end of eight months?

9. Find the discount rate if a man makes a loan from the bank for $2500 due after 8 months and receives $2387.40.

10. A asks for a loan of $3000 for 3 months from a bank charging 5% discount. What did A receive from the bank, and what rate of interest was he paying?

11. A merchant is offered the option of paying a bill in 60 days or of receiving 2% discount for cash. At what rate of interest can he afford to borrow money and get the discount? If he borrows money at 6%, how much does he save if the face of the bill is $2000?

II. Compound Interest.

Interest should be paid promptly when due or it should be added to the principal that the investor may not lose a part of his income. When interest is added to the principal to form a new principal upon which interest will be computed, the final amount is called compound amount. The difference between the compound amount and the principal originally invested is the compound interest.

The interest may be added to the principal annually, semiannually, quarterly or m times a year. The interest is said to be converted, and the interval is the conversion period.

Compound Interest Formula.

We shall find an expression for the compound amount A of any investment P, after n years with interest at rate i payable annually.

Original principal invested is	P
Interest due at end of first year	Pi
New principal at end of first year	$P + Pi = P(1+i)$
Interest due at end of second year	$P(1+i)i$
New principal at end of second year	$P(1+i) + P(1+i)$
Factoring, we have	$P(1+i)^2$
Amount at end of third year	$P(1+i)^3$
Amount at end of n years	$P(1+i)^n$

We thus have the fundamental formula of counpound interest, called the Compound Interest Law.

$$(5) \qquad A = P(1+i)^n$$

This is the formula used when the interest is converted annually. $(1+i)^n$ is called the accumulation factor. The exponent is always equal to the number of periods. The formula shows that the compound amount is proportional to the principal but it is not proportional to the time or to the rate of interest, as is simple interest.

Note. Compound amount given by formula (5) may be computed by means of the binomial theorem, by special compound amount tables, or by logarithms. We shall use the method of logarithms.

Example. Find the compound interest on \$350 for 6 years at 5%.

Solution:
$$A = 350(1.05)^6$$
$$\log A = \log 350 + 6 \log 1.05$$
$$\log 350 = 2.54407$$
$$6\log 1.05 = 6(0.02119) = 0.12714$$
$$\log A = 2.67121$$
$$A = \$469.04$$

Hence the compound interest is 469.04 - 350 = \$119.04.

Interest Converted m Times a Year.

When the interest is converted m times a year, it is customary to state the rate as an annual rate payable semi-annually, quarterly, or m times a year. Thus a note bearing interest at 6% compounded semiannually for three years means that the interest per period is 3% and that the number of periods is six. The formula becomes:

$$A = P\left(1+\frac{.06}{2}\right)^{3\times 2} = P(1.03)^6$$

The general formula is

$$(6) \qquad A = P\left(1+\frac{i}{m}\right)^{mn}$$

Where j is the annual rate and m is the number of conversions per year.

Example. Find the compound interest on $4000 for 15 years at 5% converted quarterly.

$$A = 4000\left(1+\frac{.05}{4}\right)^{15\times 4} = 4000(1.0125)^{60}$$

$$\log A = \log 4000 + 60 \log 1.0125$$

$$\log 4000 = 3.60206$$

$$60 \log 1.0125 = 60(0.00540) = 0.32400$$

$$\log A = 3.92606$$

$$A = 8434.50$$

$$C.\ I. = \$4434.50$$

Nominal and Effective Rates.

The stated annual rate is called the nominal rate and is represented by j. The rate actually realized per annum is called the effective rate and is represented by i. If the conversion is two or more times a year, the effective rate is greater than the nominal rate. Thus if the nominal rate is 6% and interest is payable semiannually, the effective rate is 0.0609 thus:

$$A = (1.03)^2 = 1.0609$$
$$i = 0.0609 \text{ or } 6.09\%.$$

The relation between the effective interest i and the nominal interest j is expressed by the formula

$$(7)\quad 1+i = \left(1+\frac{j}{m}\right)^m ; \quad i = \left(1+\frac{j}{m}\right)^m - 1$$

The symbol (j,m) means nominal rate j with m conversions per year.

Interest on Fractional Part of an Interval.

In order to find the compound amount, when there is a fractional part of a period, we first compute the compound amount for the integral number of conversions, and then accumulate this amount for the fraction of an interval at simple interest at the nominal rate.

Example. Find the amount of $500 for 2 years and 2 months at 8% converted four times a year. Amount after 2 years is $500(1.02)^8 = 585.83$. Simple interest on 585.83 for 2 months at 8% is $7.81. Final amount is 585.83 + 7.81 = $593.64. This method is slightly in favor of the creditor for

$$A = 500(1.02)^{8\ 2/3} \text{ gives } \$593.62.$$

Exercise.

1. On the same system of axes plot the graphs of the amounts of $100 at 6%, (1) simple interest, (2) compound interest. The equations are $A = 100(1 + .06t)$, and $A = 100(1.06)^t$. Which interest is greater when the time is less than one year? equal to one year? greater than one year?

2. Find the compound amount of $4280 for 7 years at 4¼%, payable annually.

3. What is the difference between the amount of $1250 due in 5 years at 5% simple interest, and the amount at 5% compound interest?

4. Calculate the interest on $10,000 at 6% per annum for ½ year by the simple interest method and by the compound interest method. Which is greater?

5. Benjamin Franklin bequeathed to the City of Philadelphia the sum of $5,000 which was to be invested at compound interest for 100 years. What was the amount at the end of the period if the average interest was 4½ per cent, payable annually?

6. Find the amount after 20 years of $1375 at 5% compounded quarterly.

7. Find the interest on $1500 at 4.5% per annum for 6 years when the interest is converted annually; when the interest is converted quarterly.

8. To what amount will $1000 accumulate in 100 years if the interest is converted semiannually at 3%? at 6%?

9. Find the amount of $5000 at 5% converted annually for four years and nine months,
   (1) By computing the compound amount for the entire time.
   (2) By finding the compound amount at the end of the fourth year, and adding to this the simple interest on this amount for nine months. Which is greater, and by how much?

10. Find the compound amount and compound interest on $2000 for 3 years 5 months at 4.1%.

11. Find the compound interest on $1600 for 3 years 4 months at 6% payable semiannually.

12. Find the amount of $100 after one year with interest at 8% compounded quarterly. What percentage is actually gained during the year due to the frequent compounding, that is, what is the effective rate?

13. A teacher has funds on deposit in a bank that pays 4% converted three times a year. He is considering the purchase of bonds which yield 4.15%. Is it to his advantage to change the form of his investment? What is the difference in annual return on $5000?

Present Value. Rate and Time.

The fundamental formula (5)

$$A = P(1+i)^n$$

has four unknowns. When any three of these are given, the fourth can always be found. To find P when A, i, and n are given, we solve for P thus,

$$P = \frac{A}{(1+i)^n} = A(1+i)^{-n}$$

This formula gives the present value P which, if invested today at rate i will accumulate to A dollars at the end of n years.

To find n or i, we proceed as follows:

$$\frac{A}{P} = (1+i)^n$$

$$\log A - \log P = n \log(1+i)$$

Then we solve for n and for $\log(1+i)$.

Exercise.

1. Find the present value of $5000 due at the end of 17 years without interest if money is worth 4%.

2. Find the present value of $2000 due 5 years from now if money is worth 4¼%.

3. At what rate will $500 amount to 608.33 in 5 years at compound interest?

4. In what length of time will a deposit of $500 amount to $800 in a savings bank that pays 4% annually?

5. At what nominal rate convertible quarterly will $100 amount to $160 in 7 years?

6. A father wishes at the birth of a son to set aside a sum that will accumulate to $2000 in 18 years when the boy enters college. If money is worth 5.5%, what is the sum required?

7. How long will it take $2000 to amount to $3000, if the interest is 4% converted semiannually?

8. The savings department of the First National Bank pays 3% interest converted every 4 months. How many years will it take $1000 to amount to $1500?

9. Find the present value of a note for $379.25 with interest payable annually at 4%, and due three years from now, if money is worth 5%. Find the present value of the note if it did not bear interest.

10. Find the present value of a note for $2500 due in 4 years with interest at 4% converted annually, if money is worth 6% payable semiannually.

11. A certain note reads as follows: "Three years and six months after date Y promises to pay X $2000". What is the value of the note one year after date if money is worth 6% converted semiannually?

12. How long will it take one dollar to double itself at 4% converted semiannually? at 6%, and at 8%, converted semiannually?

13. At what rate will a sum of money double itself in 10 years, if the interest is converted annually; semi-annually; quarterly?

Equation of Value.

An equation of value is an equation stating that the value of one set of obligations due at various dates is equal to another set of obligations due at other dates. In writing an equation of value all the obligations must be accumulated or discounted to the same date. It does not matter what date is selected, but it is often found convenient to select the present date or the final date for comparison.

The principle of the equation of value is fundamental in the mathematics of finance, and it should be thoroughly mastered.

Example. A man owes $2000 due after 10 years, $1200 due after 7 years, and $900 due in 4 years with accumulated interest at 3%. He wishes to pay in full by making two equal payments one in 3 years and one in 6 years. If money is worth 5%, find the size of the payments.

Old Obligations	New Obligations
(a) $2000 due in 10 years	$X due in 3 years
(b) 1200 due in 7 years	$X due in 6 years
(c) $900(1.03)^4$ due in 4 years	

The sum of the values of the old obligations must be equal to the sum of the values of the new obligations on the same date. If we make the comparison date 10 years from now, we have

$$2000+1200(1.05)^3+900(1.03)^4(1.05)^6 = X(1.05)^7+X(1.05)^4.$$

Solve this equation for X and thus find the required size of the payment.

Exercise.

1. A man wishes to pay a debt of $2000 due in 3 years without interest by making three equal annual payments, one at the end of each year. If money is worth 6%, find the size of each installment.

2. A man owes $1000 due in 1 year, $1000 due in 2 years, and $1000 due in 3 years. If he wishes to discharge all his obligations now, what sum should he pay, money being worth 4%?

3. A man wants to sell a house. He is offered $3000 cash and $2400 at the end of each year for three years, or $4000 cash and $2000 at the end of each year for three years. If money is worth 7%, which is the better offer?

4. A man owes $6 due in 2 years, $400 due in 3 years, and $700 due in 5 years and wishes to know what single payment would discharge all these debts at the end of 1 year, if money is worth 5%.

5. On what single date could the sum of the obligations in exercise (4) (i. e. $1700) be paid as an equitable settlement of the debt?

6. Two bills for $480 and $650 without interest come due in 4 months and 9 months respectively. Find the sum that will be required to pay both bills at the end of 6 months, if money is worth 8%.

7. A man owes $1000 due in 2 years, $800 due in 3 years, and $500 with interest at 6% due in 4 years. He wishes to pay these debts in two equal payments made at the end of one year and at the end of three years. Find the size of the payments, if money is worth 5%.

III. Annuities.

An annuity is a series of equal payments made at equal intervals of time. Pensions and insurance premiums are examples of annuities.

In an ordinary annuity the payments are made at the end of the intervals of time. If the payments are made at the beginning of the intervals, the annuity is called an annuity due. The total annual payment is called the annual rent.

Amount of an Annuity.

The sum to which the entire payments accumulate is called the amount of the annuity. The amount of an annuity of 1 per year at the end of n years at rate i per year is denoted by $s_{\overline{n}|}$ (s sub n). We shall now find an expression for $s_{\overline{n}|}$ in terms of i and n.

It is evident that the income of 1 at rate i is an ordinary annuity the annual rent of which is i. But the amount of this annuity at the end of n years is equal to the compound interest on 1 for n years, which is $(1+i)^n - 1$. If we divide this expression by i, the quotient is the amount of an annuity the annual rent of which is 1 instead of i. This is true because the amount is proportional to the rent. Thus, we have the formula for the amount of an annuity of 1:

$$(9) \qquad s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

If the annual payment is R instead of 1, and if we designate the amount by S, we have

$$(10) \qquad S = Rs_{\overline{n}|} = R\,\frac{(1+i)^n - 1}{i}$$

This equation has four variables, S, R, i, and n. If any three of these are known the value of the fourth is fixed and can be found.

Formulas (9) and (10) are true when the interest is converted once a year. If the interest is converted m times a year, we replace

$(1+i)$ by $\left(1+\frac{i}{m}\right)^m$ and i by $\left(1+\frac{i}{m}\right)^m - 1$.

Example. Find the amount of an annuity of $100 a year for seven years, if the interest is 4% converted annually.

$$S = 100\,\frac{(1.04)^7 - 1}{.04}$$

$$\log(1.04)^7 = 7(0.01703) = 0.11921$$

$$(1.04)^7 = 1.31585$$

$$S = 100\,\frac{0.31585}{.04} = \frac{3158.5}{4}$$

$$S = \$789.63$$

Note. If a seven place table is used, S = 789.83.

If the interest is 4% payable semiannually, we have

$$S = 100\,\frac{(1.02)^{14} - 1}{.0404} = \$790.79$$

Exercise.

1. A man sets aside $100 at the end of each year towards a fund for his son's college expenses. If the interest is 5% what sum will be available at the end of 18 years?

2. Find the amount of an annuity of $150 per year payable annually for 20 years, if interest is 6% converted semiannually?

3. What sum must be deposited annually in a savings bank paying 3% interest converted every four months to provide for the purchase of a car worth $1250 in a period of seven years?

4. Show that $S_{\overline{n}|} = (1+i)\,s_{\overline{n}|}$ is the correct formula for the amount of an annuity due.

5. Sometimes the annual rent R is made in p equal installments. If an annuity of $500 a year for 6 years payable semiannually, the interest being 4% a year payable semiannually, show that it will have the same final amount as an annuity of $250 a year for 12 years at 2%.

6. What sum must be deposited in the bank at the end of each year to amount to $2000 at the end of 10 years, if the interest is 4% converted annually?

Present Value of An Annuity.

Insurance companies often wish to find the cash value of a certain annuity. The present value of an annuity is equal

to the sum of the present values of all the individual payments of the annuity. This means that if the present value of an annuity is put at compound interest for the time and rate of the annuity, its amount will be equal to the amount of the annuity. If we denote the present value of the annuity by A, its amount will be represented by $A(1+i)^n$, and since the amount of the annuity is $R\frac{(1+i)^n - 1}{i}$ by formula (10), these two expressions are equal. Thus,

$$A(1+i)^n = R\frac{(1+i)^n - 1}{i}. \qquad \text{Hence}$$

$$(11) \qquad A = R\frac{1-(1+i)^{-n}}{i}.$$

This formula may be used whenever the payment interval of the annuity equals the conversion period of the interest.

In order to find the annuity that a certain sum of money would purchase, we solve formula (11) for R.

Exercise.

1. What is the present worth of $400 a year for 15 years at 4% interest converted annually?

2. Find the present value and the amount of an annuity of $300 a year payable annually for 50 years, if money is worth 6%.

3. A man pays a debt of $3250 with interest at 5% in six equal annual installments. Find the amount of each payment.

4. If a $100,000 issue of bonds is to be redeemed in 20 years and money can be accumulated at 4% compound interest, what sum must be raised annually to meet the cost?

5. If the bonds in exercise (4) bear interest at 5%, how much must be raised annually to liquidate the debt?

6. A man bought a house for $8000. He paid $3000 cash and agreed to pay the balance in 10 equal installments including interest at 6%. Find the annual payment. (Here we have given the present value of an annuity, and it is required to find the annual rent).

7. A house is offered for sale for $5000, or for $1000 cash and $500 at the end of each year for 10 years. Which is the better plan for the buyer, if money is worth 5%?

END OF YEAR